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NUMERICAL APPLICATION OF  
ONE NEW APPROXIMATE METHOD  
FOR SOLVING BOUNDARY VALUE PROBLEMS

*by M. A. Aleksidze, N. M. Arveladze,  
and N. L. Lekishvili*

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## ANNOTATION

This monograph analyzes the questions involved in the numerical application of one new method for solving boundary value problems — the method of functional equations — and derives the proper general purpose programs.



## TABLE OF CONTENTS

	<u>PAGE</u>
ANNOTATION	iii
INTRODUCTION	1
§ 1. Dirichlet Problem and Green Formulas . . . . .	2
§ 2. Fundamental Formula for the Theory of Harmonic Functions . . .	7
§ 3. Some Concepts of Functional Analysis . . . . .	16
§ 4. Linear Independence and Completeness of Several Systems of Harmonic Functions . . . . .	27
§ 5. Approximate Method for Solving the Dirichlet Problems . . .	35
§ 6. Solution to Boundary Value Problems with the Aid of Nonorthogonal Series . . . . .	53
§ 7. Series of Nonorthogonal Systems of Functions . . . . .	65
§ 8. Nonorthogonal Series in Variational Methods . . . . .	89
§ 9. Approximate Solution to One Mixed Boundary Value Problem in the Theory of Harmonic Functions . . . . .	95
§ 10. Approximate Solution to the Reimann-Hilbert Problem . . . .	104
§ 11. Solution to the External Dirichlet Problem for the Laplace Equation Using the Method of Functional Equations . . .	120
§ 12. Solution to the Two-Dimensional External Dirichlet Problem by the Method of Generalized Fourier Series . . . . .	151
REFERENCES	187

11/11/11

11/11/11

11/11/11

11/11/11

11/11/11

11/11/11

11/11/11

11/11/11

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## INTRODUCTION

Many important problems in natural science can be reduced to so-called boundary value problems. Development of general methods for solving these problems and investigation of the total automation of their approximate solution on general purpose computers are problems of paramount significance. In the Department of Numerical Methods of the Computer Center, Academy of Sciences, Georgian SSSR under the direction of V.D. Kupradze, a general approximate method was developed for solving boundary value problems. In the present publication this method is discussed relative to the first internal boundary value problem of the theory of harmonic functions. The general purpose programs cited at the end of this publication make it possible to find solutions to the external Dirichlet boundary value problems for ellipsoidal regions on the high speed electronic computer BESM-2. These problems arise in computing gravity, magnetic, electric and heat fields, in solving problems in hydrodynamics of an ideal fluid, in solving certain problems in the theory of elasticity, etc.

In 1961, at a seminar held at the Computer Center of the Academy of Sciences, Georgian SSR, V. D. Kupradze indicated the possibility of finding an approximate solution to boundary value problems with the aid of integral Expressions (5.1) and (5.2) by replacing them with quadrature formulas. Solution to numerous examples carried out by N.A. Papunashvili showed that this approach to the approximate solution of boundary value problems (in Reference [3], it was called the second approach) leads to the very poorly stipulated systems of linear equations. In several instances an increase in the number of points in the quadrature formula leads to a deterioration of the final results. This is due to the fact that (5.2) is a functional equation of the first sort, and that its approximate solution is an improper problem. A

special seminar was held to discuss the questions of justifying and approving this method; at this seminar a new method was developed and justified for an approximate solution to the boundary value problems that is also based on Expressions (5.1) and (5.2) (Reference [3], calls this the first approach). The results obtained at this seminar are discussed in Reference [1-3].

Initially the present publication was regarded as a collection of general-purpose programs, instructions for them and numerical examples, solved approximately using these general purpose programs. Then part of the general purpose programs for solving the internal Dirichlet problem was published in /6 Moscow ("Giprotis")\* [27,28], which made it possible to give a more complete discussion of the approximate method and to define certain concepts as well as the proof and conclusions of known theorems and formulas, which were used in demonstrating the approximate method. It seems to us that this will facilitate understanding of the material discussed.

#### § 1. Dirichlet Problem and Green Formulas

Let  $V$  be a finite region bounded by a closed Lyapunov surface  $S$ . The internal Dirichlet problem is as follows: We seek a harmonic function  $u$  in the region  $V$ , i.e.,

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0, \quad (1.1)$$

which at the boundary  $s$  assumes specified values  $\psi(s)$ :

$$u|_s = \psi(s). \quad (1.2)$$

The external infinite region with the boundary  $s$  is denoted by  $V_e$ . The external Dirichlet problem is as follows. We seek a harmonic function in  $V_e$  which at the boundary  $s$  satisfies Expression (1.2).

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\* Translator's note: State Institute for Standard Experimental Planning and Technical Research.

The method discussed below for solving the boundary value problems is based on several integral expressions. For the Dirichlet and Neumann boundary value problems (both interior and exterior) such an expression is called the Green formula.

First let us find the Gauss divergence theorem which establishes the relationship between the triple integral over the volume  $V$  and the integral over the surface  $S$  which bounds this area [9].

Let  $A_i(x)$ ,  $i = 1, 2, 3$ , be functions which have continuous first derivatives in the region  $V$ . We can use the following rule [9] for computing the triple integrals.

For reduction of the triple integral

$$\iiint_V f(x_1, x_2, x_3) dV$$

to a single and double integral: (1) let us map the surface  $S$ , which bounds the region  $V$ , onto the plane  $x_1, x_2$  in the form of the region  $\delta$ ; (2) let us determine the coordinates  $x_3^{(1)}$  and  $x_3^{(2)}$  of the points of entry and exit of the straight line, parallel to the axis  $Ox_3$  and passing through the point  $(x_1, x_2)$  of the region  $\delta$ ; (3) if we assume  $x_1, x_2$  to be constants, we can compute the integral

$$\int_{x_3^{(1)}}^{x_3^{(2)}} f(x_1, x_2, x_3) dx_3,$$

and then the double integral

$$\iint_{\delta} d\sigma \int_{x_3^{(1)}}^{x_3^{(2)}} f(x_1, x_2, x_3) dx_3.$$

Applying this rule to the function  $\frac{\partial A_i}{\partial x_i}$  and recalling that the integral of the derivative is equal to the difference in values of the primitive function at the upper and lower limits, we obtain

$$\iiint_V \frac{\partial A_1}{\partial x_1} dV = \iint_{\tilde{S}} d\sigma \int_{x_3^{(1)}}^{x_3^{(2)}} \frac{\partial A_1}{\partial x_1} dx_3 = \iint_{\tilde{S}} [A_1(x_1, x_2, x_3^{(2)}) - A_1(x_1, x_2, x_3^{(1)})] d\sigma.$$

Let us divide the surface  $S$  into three parts:  $S_1$  is the set of points of entry into the region  $V$  of the straight lines parallel to the axis  $Ox_1$ ;  $S_2$  is the set of points of exit from the region  $V$  of the straight lines parallel to the axis  $Ox_1$ ;  $S_3$  is the set of points belonging to the parallel axis  $Ox_1$ , tangent to  $S$ . Let us denote by  $(n_x, x_1)$  the angle between the axis  $Ox_1$  and the outer normal  $n_x$  to  $S$  at the point  $x \in S$ .

From the trivial equations

$$d\sigma = \cos(n_x, x_1) ds \text{ for } S_1, \quad d\sigma = -\cos(n_x, x_1) ds \text{ for } S_2,$$

where  $ds$  is the element of  $S$  (area of an infinitely small vicinity of the point  $x \in S$  for  $S$ ), we find

$$\iiint_V \frac{\partial A_1}{\partial x_1} dV = \iint_{S_2} A_1(x_1, x_2, x_3^{(2)}) \cos(n_x, x_1) ds + \int_{S_1} A_1(x_1, x_2, x_3^{(1)}) \cos(n_x, x_1) ds,$$

or taking the fact into account that at the points  $S_3$

$$\cos(n_x, x_1) = 0,$$

we find (the index  $x$  at the normal will be dropped in the future)

$$\iiint_V \frac{\partial A_1}{\partial x_1} dV = \iint_S A_1 \cos(n, x_1) ds.$$

After writing analogous equations for the functions  $A_2$  and  $A_3$  and combining them, we find ultimately the Gauss divergence theorem

$$\iiint_V \sum_{i=1}^3 \frac{\partial A_i}{\partial x_i} dV = \iint_S \sum_{i=1}^3 A_i \cos(n, x_i) ds. \quad (1.3)$$

Let us note that Formula (1.3) is valid also for those regions whose boundaries  $s$  contain individual lines with points which have no normal  $n$  [4]. However, the measure (area) of the set of such points must be equal to zero. As a result, the exclusion of these points will not influence the value of the 1/8 limit to which the integral sums tend. In practice the boundary  $s$  must consist of a finite number of surfaces such that a normal  $n$  exists for each interior point.

Let us analyze the adjoint linear differential operations

$$Lu = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + \sum_{i=1}^3 b_i \frac{\partial u}{\partial x_i} + cu,$$

$$L^*u = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} - \sum_{i=1}^3 \frac{\partial (b_i u)}{\partial x_i} + cu.$$

It is not difficult to prove directly that they satisfy the following equations:

$$vLu - uL^*v = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial}{\partial x_j} a_{ij} \left( v \frac{\partial u}{\partial x_i} - u \frac{\partial v}{\partial x_i} \right) + \sum_{i=1}^3 \frac{\partial}{\partial x_i} (b_i uv). \quad (1.4)$$

If we assume that  $u$  and  $v$  have continuous derivatives up to second order inclusively, by integrating (1.4) and taking Formula (1.3) into account we obtain

$$\iiint_V vLu - uL^*v \, dV = \iint_s \left[ \sum_{i=1}^3 \sum_{j=1}^3 n_j a_{ij} \left( v \frac{\partial u}{\partial x_i} - u \frac{\partial v}{\partial x_i} \right) + \sum_{i=1}^3 b_i n_i uv \right] ds. \quad (1.5)$$

Expression (1.5) is called the Green formula. Since we obtain from it the so-called fundamental formula of the theory of harmonic functions, then it is important to know in which instances it remains valid. Let the functions  $u$  and  $v$  have integrable derivatives of second order which are continuous only inside the region  $V$ . Thus, for example, when region  $V$  approaches the boundary  $s$ , the second derivatives of the functions  $u$  and  $v$  may increase

without bound, undergoing an infinite discontinuity at the boundary points. It is easy to show [4] that in this case the Green formula remains valid. Let us look at the region  $V'$  which is contained inside the region  $V$  along with its own boundary  $s'$ . Since the left-hand side of Formula (1.5) is integrable, then when  $V' \rightarrow V$  the limit of the integral of  $V'$  does not depend on the way that  $V'$  approaches  $V$  and by definition is an integral over the region  $V$ . The expression under the sign of the integral in the right-hand side of the Green formula is continuous in the region  $V$  up to its boundary. Therefore, when  $V' \rightarrow V$  the integral of this expression at the boundary  $s'$  of the region  $V'$  varies continuously and converges to a limit which may be only the integral over  $s$ . But as long as  $V' \neq V$ , Formula (1.5) is valid; consequently, when  $V' \rightarrow V$  its left- and right-hand sides approach the same limit.

When  $a_{ij} = 1$  ( $i, j = 1, 2, 3$ ) and  $b_i = 0$  ( $i = 1, 2, 3$ ), if we take into account the fact that the expression

$$\sum_{i=1}^3 \sum_{j=1}^3 n_j a_{ij} \frac{\partial}{\partial x_i} = \sum_{j=1}^3 n_j \frac{\partial}{\partial x_j} \equiv \frac{d}{dn} \quad /9$$

in this case represents the differentiation operator in the direction of the outer normal  $n$  to  $s$ , for the Green formula we find

$$\iiint_V (v \Delta u - u \Delta v) dV = \iint_s \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds. \quad (1.6)$$

In the two-dimensional case, Formulas (1.5) and (1.6) assume, respectively, the following form:

$$\iiint_V (v Lu - u L^* v) dV = \int_s \left[ \sum_{i=1}^2 \sum_{j=1}^2 n_j a_{ij} \left( v \frac{\partial u}{\partial x_i} - u \frac{\partial v}{\partial x_i} \right) + \sum_{i=1}^2 b_i n_i u v \right] ds, \quad (1.7)$$

$$\iiint_V (v \Delta u - u \Delta v) dV = \int_s \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds. \quad (1.8)$$

## § 2. Fundamental Formula for the Theory of Harmonic Functions.

Let us analyze the function

$$\frac{1}{r} = \frac{1}{\sqrt{\sum_{i=1}^3 (y_i - x_i)^2}},$$

where  $y_i$  and  $x_i$  ( $i = 1, 2, 3$ ) are coordinates of the two points  $y$  and  $x$ .

We can show that when  $y \neq x$  it satisfies the Laplace equation

$$\Delta \frac{1}{r} = 0.$$

In fact, the following expressions are valid:

$$\frac{\partial}{\partial x_i} \frac{1}{r} = \frac{x_i - y_i}{r^3}, \quad \frac{\partial^2}{\partial x_i^2} \frac{1}{r} = -3 \frac{(x_i - y_i)^2}{r^5} + \frac{1}{r^3},$$

from which we find that

$$\Delta \frac{1}{r} = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} \frac{1}{r} = \frac{3}{r^3} - \frac{3}{r^5} \sum_{i=1}^3 (x_i - y_i)^2 = 0.$$

The function

$$\Gamma(x, y) = \frac{1}{4\pi} \left[ \frac{1}{r(x, y)} + \varphi(x, y) \right],$$

where  $\varphi(x, y)$  is harmonic with respect to  $y$  and continuous along with its first derivatives in the region  $V$ , is called the fundamental solution to the Laplace equation in the region  $V$ . Let  $x$  not belong to the bounded region  $V$ . Then, the fundamental solution  $\Gamma(x, y)$  is harmonic in this region as a result of /10  
which, after substitution into the Green Formula (1.6),

$$v(y) = \Gamma(x, y),$$

we find

$$\iiint_V \Gamma(x, y) \Delta u(y) V dy = \iint_s \left[ \Gamma(x, y) \frac{du(y)}{dn_y} - u \frac{d\Gamma(x, y)}{dn_y} \right] ds_y, \quad x \in V_e.$$

This latter equation for the harmonic function  $u$  assumes the following form (below we shall drop the indices at the normal and the variable of integration):

$$\iint_s \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds = 0, \quad x \in V_e. \quad (2.1)$$

Let us analyze the case when the point  $x$  lies inside the region  $V$  and denote by  $W_\varepsilon$  a sphere with an arbitrarily small radius  $\varepsilon$  with the center at the point  $x$ , lying completely in the region  $V$ . Using the Green Formula (1.6) in the region  $V - W_\varepsilon$ , we obtain

$$\iint_s \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds = \iiint_{V-W_\varepsilon} \Gamma \Delta u dV - \iint_{W_1} \Gamma \frac{du}{dn} ds + \iint_{W_1} u \frac{d\Gamma}{dn} ds, \quad (2.2)$$

where  $W_1$  is the surface of the sphere  $W_\varepsilon$ . With regard to the expression for the fundamental function  $\Gamma$ , the third integral in the right-hand side of this latter equation assumes the following form:

$$\iint_{W_1} u \frac{d\Gamma}{dn} ds = \frac{1}{4\pi} \iint_{W_1} u \frac{d\varphi}{dn} ds + \frac{1}{4\pi} \iint_{W_1} u \frac{d}{dn} \left( \frac{1}{r} \right) ds.$$

Taking into account the fact that on the spherical surface  $W_1$   $\frac{d}{dn} = -\frac{d}{dr}$  ( $n$  is the outer normal),  $r = \varepsilon$ , and that the first integral in the right-hand side of the last equation vanishes when  $\varepsilon \rightarrow 0$ , we find

$$\iint_{W_1} u \frac{d\Gamma}{dn} ds = \frac{1}{4\pi\varepsilon^2} \iint_{W_1} u ds,$$

or by using the mean value theorem and the equation

$$\iint_{W_1} ds = 4\pi\varepsilon^2,$$

we ultimately find the following asymptotic equation:

$$\lim_{\varepsilon \rightarrow 0} \iint_{W_1} u \frac{d\Gamma}{dn} ds = \lim_{\varepsilon \rightarrow 0} \frac{u_{av}}{4\pi\varepsilon^2} \iint_{W_1} ds = \lim_{\varepsilon \rightarrow 0} u_{av} = u(x), \quad (2.3)$$

where  $u_{av}$  is the value of the function  $u$  at a certain point belonging to the sphere  $W_\varepsilon$ .

The first integral in the right-hand side of Expression (2.2), when  $\varepsilon \rightarrow 0$  /11  
approaches the improper integral

$$\lim_{\varepsilon \rightarrow 0} \iiint_{V-W_\varepsilon} \Gamma \Delta u dV - \iiint_V \Gamma \Delta u dV, \quad (2.4)$$

if this latter exists, and the second integral in the right-hand side of (2.2) vanishes when  $\varepsilon \rightarrow 0$

$$\lim_{\varepsilon \rightarrow 0} \iint_{W_1} \Gamma \frac{du}{dn} ds = 0 \quad (2.5)$$

since the derivative  $\frac{du}{dn}$  is continuous (based on the assumption used in deriving the Green formula) and consequently is bounded, and the function  $\Gamma$  increases when  $\varepsilon \rightarrow 0$  on  $W_1$  as  $1/\varepsilon$ , whereas the area of the surface  $W_1$  decreases as  $\varepsilon^2$ .

Substituting Equations (2.3) - (2.5) into (2.2) we find

$$\iint_s \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds = \iiint_V \Gamma \Delta u dV + u(x), \quad x \in V - s.$$

This latter equation for the harmonic function  $u$  assumes the following form:

$$\iint_s \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds = u(x), \quad x \in V - s. \quad (2.6)$$

Let us finally analyze the case when the point  $x$  is located at the boundary of the surface  $S$ . Let us denote by  $W_\varepsilon'$  the part of the sphere  $W_\varepsilon$  which lies in the region  $V$  and use the Green Formula (1.6) in the region  $V - W_\varepsilon'$

$$\iint_{s=W_2} \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds = \iiint_{V=W_\epsilon} \Gamma \Delta u dV - \iint_{W_3} \Gamma \frac{du}{dn} ds + \iint_{W_3} u \frac{d\Gamma}{dn} ds, \quad (2.7)$$

where  $W_2$  is the part of the boundary  $s$ , lying in the sphere  $W_\epsilon$ ;  $W_3$  is the part of the surface of the sphere  $W_\epsilon$  lying in the region  $V$ .

Repeating all the arguments of the previous case, we must have the value of the integral  $\iint_{W_3} ds$ , equal to the area of the part of the surface

of the sphere  $W_\epsilon$  which lies in the region  $V$ . To compute this integral, let us introduce at point  $x$  a local system of coordinates  $\xi_1, \xi_2, \xi_3$ , with  $\xi_3$  directed along the outer normal to the surface  $S$  at point  $x$ . We shall assume that the equation for the surface  $S$  inside the sphere  $W_\epsilon$  can be written in the form

$$\xi_3 = f(\xi_1, \xi_2), \quad (2.8)$$

where the function  $f$  and its first-order derivatives are continuous and vanish at point  $x$ . We can show [4] that any sufficiently smooth surface can be described in the form of (2.8) in a sphere of sufficiently small radius. /12  
Expanding the function  $f(\xi_1, \xi_2)$  in a Taylor series in the immediate vicinity of point  $x$  and taking into account all first-order terms, we find the following relationship

$$\xi_3 = f'_{\xi_1}(\bar{\xi}_1, \bar{\xi}_2) \tilde{\xi}_1 + f'_{\xi_2}(\bar{\xi}_1, \bar{\xi}_2) \tilde{\xi}_2, \quad (2.9)$$

where  $|\bar{\xi}_i| < |\tilde{\xi}_i| < |\tilde{\xi}_i|$ ,  $f'_{\xi_i}(\bar{\xi}_1, \bar{\xi}_2)$  ( $i=1, 2$ ) are the values of the derivative of the function  $f$  with respect to variable  $\xi_i$  at the point  $(\bar{\xi}_1, \bar{\xi}_2)$  and on the strength of the above, they vanish simultaneously with  $\bar{\xi}_1, \bar{\xi}_2$ . Substituting into (2.9) the values  $\xi_i$  ( $i=1, 2, 3$ ), expressed with the aid of the spherical coordinates

$$\xi_1 = r \sin \theta \cos \varphi, \quad \xi_2 = r \sin \theta \sin \varphi, \quad \xi_3 = r \cos \theta,$$

we find

$$\cos \theta = f'_{\xi_1} \sin \theta \cos \varphi + f'_{\xi_2} \sin \theta \sin \varphi = h(r, \theta, \varphi),$$

where  $h$  is a function which is bounded and vanishes simultaneously with  $r$ , and  $\theta$  is an angular coordinate of the point on the surface  $S$ . Now let us concern ourselves with computing the integral that is of interest to us

$$\begin{aligned} \frac{1}{4\pi\varepsilon^2} \iint_{W_3} ds &= \frac{1}{4\pi\varepsilon^2} \iint_{W_3} r^2 \sin \theta d\theta d\varphi = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\bar{\theta}} \sin \theta d\theta = \\ &= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin \theta d\theta + \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{\pi/2}^{\bar{\theta}} \sin \theta d\theta = \frac{1}{2} + \frac{1}{4\pi} \int_0^{2\pi} d\varphi \left[ -\cos \theta \right]_{\pi/2}^{\bar{\theta}} = \\ &= \frac{1}{2} + \frac{1}{4\pi} \int_0^{2\pi} h(\varepsilon, \bar{\theta}, \varphi) d\varphi = \frac{1}{2} + H(\varepsilon), \end{aligned} \quad (2.10)$$

where

$$H(\varepsilon) = \frac{1}{4\pi} \int_0^{2\pi} h(\varepsilon, \bar{\theta}, \varphi) d\varphi$$

is a bounded function which vanishes simultaneously with  $\varepsilon$ . Taking (2.10) into account and using the mean value theorem for the third integral in the right-hand side of (2.7) we find the following asymptotic equation:

$$\lim_{\varepsilon \rightarrow 0} \iint_{W_3} u \frac{d\Gamma}{dn} ds = \lim_{\varepsilon \rightarrow 0} \frac{u_{cp}}{4\pi\varepsilon^2} \iint_{W_3} ds = \lim_{\varepsilon \rightarrow 0} u_{cp} \left[ \frac{1}{2} + H(\varepsilon) \right] = \frac{u(x)}{2}.$$

Applying the arguments of the previous case ( $s \in V-S$ ), to the other two integrals in the right-hand side of Expression (2.7), we find the relationship

$$\iint_V \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds = \iiint_V \Gamma \Delta u dV + \frac{u(x)}{2}, \quad x \in S,$$

which for the harmonic function  $u$  takes the form

$$\iint_S \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds = \frac{u(x)}{2}, \quad x \in S. \quad (2.11)$$

/13

After combining Formulas (2.1), (2.6) and (2.11) into one, we find the basic formula for the theory of harmonic functions

$$\iint_s \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds = \begin{cases} 0 & \text{when } x \in V_e \\ \frac{1}{2} u(x) & \text{when } x \in S \\ u(x) & \text{when } x \in V - S. \end{cases} \quad (2.12)$$

Formula (2.12) remains in force if  $V$  is an infinite region with a finite boundary  $s$ . For this let us analyze the sphere  $W$  of finite radius  $\rho$ , containing the boundary  $s$  inside itself. The intersection (part) of the regions  $V$  and  $W$  can be denoted by  $V^*$ . After using Formula (2.12) in the region  $V^*$ , we arrive at a formula whose left-hand side will differ in form from the left-hand side of Formula (2.12) in that the integral

$$\iint_{W_4} \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds, \quad (2.13)$$

is added to it, where  $W_4$  is the surface  $W$  of the sphere.

To prove that this integral equals zero, we must know the asymptotic behavior of the harmonic functions at infinity [4]: the function  $u$ , which is harmonic in the infinite region, satisfies the inequality

$$|u(x)| < \frac{A}{r}, \quad \left| \frac{\partial u}{\partial x_i} \right| < \frac{A}{r^2} \quad (i=1, 2, 3), \quad (2.14)$$

where

$$r = \sqrt{\sum_{i=1}^3 x_i^2}.$$

Inequality (2.14) is a simple corollary of Kelvin's theorem [4]. Taking (2.14) into account and determining the fundamental solution, we conclude that when  $\rho \rightarrow \infty$  the integrand in (2.13) decreases as  $1/\rho^3$ , whereas the area of the surface  $W_4$  of the sphere  $W$  increases as  $\rho^2$ . Passing to the limit when  $\rho \rightarrow \infty$ , we again find Formula (2.12).

In the two-dimensional case, the function

$$\Gamma(x, y) = -\frac{1}{2\pi} \left[ \ln \frac{1}{r(x, y)} + \varphi(x, y) \right], \quad (2.15)$$

is called the fundamental solution to the Laplace equation, where  $r(x, y)$  is the distance between points  $x$  and  $y$  on the plane, and  $\varphi(x, y)$  is a function which is harmonic in the two-dimensional region  $V$  with respect to the coordinates of the point  $y$ . It is easy to prove that when  $x \neq y$  the function  $\ln 1/r$  is /14 harmonic for the coordinates of points  $x$  and  $y$ .

The fundamental formula of the theory of harmonic functions for the two-dimensional bounded region  $V$  has the following form:

$$\int_s \left( \Gamma \frac{du}{dn} - u \frac{d\Gamma}{dn} \right) ds = \begin{cases} 0 & \text{when } x \in V_e \\ \frac{1}{2} u(x) & \text{when } x \in s. \\ u(x) & \text{when } x \in V-s. \end{cases} \quad (2.16)$$

It is derived completely in analogy with Equation (2.12), and therefore, we shall not give its derivation.

In the two-dimensional case, we must especially analyze [4] the case of the infinite region with a finite boundary  $s$ . Let us look at the finite region  $V_1$ , included between the boundaries  $S$  and  $C$ , where  $C$  is a circle of radius  $a$  with the center at the origin, encompassing  $S$ . Let us use Formula (2.6) in the region  $V_1$

$$\frac{1}{2\pi} \int_s \left( \frac{du}{dn} \ln \frac{1}{r} - u \frac{d}{dn} \ln \frac{1}{r} \right) ds + \frac{1}{2\pi} \int_C \left( \frac{du}{dn} \ln \frac{1}{r} - u \frac{d}{dn} \ln \frac{1}{r} \right) ds = \quad (2.17)$$

$$= \begin{cases} 0 & \text{when } x \in R - V_1 \\ \frac{1}{2} u(x) & \text{when } x \in S + C \\ u(x) & \text{when } x \in V, \end{cases}$$

where  $R$  is the entire plane.

If we assume that  $\frac{du}{dn} \ln \frac{1}{r}$  vanishes no more slowly than

$$\frac{\ln(x_1^2 + x_2^2)}{x_1^2 + x_2^2},$$

we find that

$$\lim_{a \rightarrow \infty} \int_C \frac{du}{dn} \ln \frac{1}{r} ds = 0. \quad (2.18)$$

Since on the circle  $C$   $\frac{d}{dn} \ln \frac{1}{r} = -\frac{1}{a}$ , then the integral

$$-\frac{1}{2\pi} \int_C u \frac{d}{dn} \ln \frac{1}{r} ds$$

approaches the "mean value of the function at infinity"

$$u_\infty = \lim_{a \rightarrow \infty} \frac{1}{2\pi a} \int_C u ds. \quad (2.19)$$

as  $a \rightarrow \infty$ .

Substituting Formulas (2.18) and (2.19) into (2.17), we find the fundamental formula for the harmonic functions in the infinite two-dimensional region

$$\frac{1}{2\pi} \int_s \left( \frac{du}{dn} \ln \frac{1}{r} - u \frac{d}{dn} \ln \frac{1}{r} \right) ds + u_\infty = \begin{cases} 0 & \text{when } x \in R_2 - V \\ \frac{1}{2} u(x) & \text{when } x \in s \\ u(x) & \text{when } x \in V. \end{cases} \quad (2.20) \quad \underline{/15}$$

Substituting into Formula (2.12) the values

$$u=1, \quad \Gamma = \frac{1}{4\pi} \frac{1}{r},$$

we find the Gauss formula

$$\iint_s \frac{d}{dn} \frac{1}{r} ds = \begin{cases} -4\pi & \text{when } x \in V \\ -2\pi & \text{when } x \in s \\ 0 & \text{when } x \in V_e. \end{cases} \quad (2.21)$$

In the two-dimensional case, the Gauss formula has the form

$$\int_s \frac{d}{dn} \ln \frac{1}{r} ds = \int_s \frac{\cos \varphi}{r} ds = \begin{cases} -2\pi & \text{when } x \in V \\ -\pi & \text{when } x \in s \\ 0 & \text{when } x \in V_e, \end{cases} \quad (2.22)$$

where  $\phi$  is the angle between directions  $n$  and  $r$ .

Let us denote by

$$\frac{du(\xi)}{dn_e}, \frac{du(\xi)}{dn_i}$$

the limiting values of the derivatives of  $u(x) = \iint_S \frac{\rho}{r} ds$  in the direction of the normal when the point  $x$  approaches  $\xi \in S$  from outside  $S$  and from inside  $S$  respectively and by  $du/dn_0$  the value of the derivative with respect to the normal of the expression for the potential of a single layer (2.23) at the point  $x = \xi \in S$ .

The values  $du(\xi)/dn_e$  and  $du(\xi)/dn_i$  are called [4], respectively, the outer and inner normal derivative of the potential of a single layer at the point  $\xi$ , and the value  $du/dn_0$  is the true value of the normal derivative at this same point. If we use the continuity at point  $\xi$  of the Expression [4]

$$\iint_S \rho \frac{d}{dn_0} \frac{1}{r} ds - \rho_0 \iint_S \frac{d}{dn} \frac{1}{r} ds,$$

where  $d/dn_0$  is differentiation with respect to the outer normal to  $s$  at point  $\xi$ ,  $d/dn$  is differentiation with respect to the outer normal at a variable point of the surface  $S$ ,  $\rho_0 = \rho(\xi)$  and taking into account the Gauss Formula (2.21), we find

$$\begin{aligned} \frac{du(\xi)}{dn_e} &= \frac{du(\xi)}{dn_0} - 2\pi\rho(\xi), \\ \frac{du(\xi)}{dn_i} &= \frac{du(\xi)}{dn_0} + 2\pi\rho(\xi). \end{aligned} \tag{2.24} \quad /16$$

In the two-dimensional case, the analogous relationships have the form

$$\begin{aligned} \frac{du(\xi)}{dn_e} &= \frac{du(\xi)}{dn_0} - \pi\rho(\xi), \\ \frac{du(\xi)}{dn_i} &= \frac{du(\xi)}{dn_0} + \pi\rho(\xi). \end{aligned} \tag{2.25}$$

### § 3. Some Concepts of Functional Analysis

Let us introduce some concepts [7] of functional analysis, which will be used in the future.

The set  $R$  of elements  $x, y, z, \dots$  is termed linear, if in it we define the operations of addition, denoted by a "+" sign and multiplication by numbers (real or complex), which do not go beyond the limits of  $R$  and which satisfy the following conditions:

1. Addition is associative, i.e.,  $(x + y) + z = x + (y + z)$ .
2. There exists a zero element  $0$  such that  $x + 0 = 0 + x = x$  for any  $x \in R$ .
3. Addition is commutative:  $x + y = y + x$ .
4.  $(\alpha + \mu)x = \alpha x + \mu x$ .
5.  $\alpha(x + y) = \alpha x + \alpha y$ .
6.  $\alpha(\mu x) = (\alpha\mu)x$ .
7.  $1 \cdot x = x$ .

Here the Latin letters denote elements of  $R$ , and the Greek letters denote numbers.

We shall say that scalar product is defined in the linear set  $R$  if corresponding to each pair of its elements  $x$  and  $y$ , taken in a given order, there is a complex number  $(x, y)$ , this number is called the scalar product of these elements and satisfies the following conditions:

1. The scalar products  $(x, y)$  and  $(y, x)$  are complex conjugate numbers

$$(x, y) = \overline{(y, x)}.$$

2. For any elements  $x, y, z \in R$  and any complex numbers  $\alpha_1$  and  $\alpha_2$  the following equation is valid

$$(\alpha_1 x + \alpha_2 y, z) = \alpha_1 (x, z) + \alpha_2 (y, z).$$

3. The scalar product of the element  $x$  by itself is a nonnegative number, equal to zero only when  $x = 0$ , i.e.,  $(x, x) \geq 0$ .

The set  $R$  is called a metric space if for any two of its elements  $x$  and  $y$  /17  
the concept of distance  $\rho(x, y)$  is defined to satisfy the following conditions:

1.  $\rho(x, y) \geq 0$  and  $\rho(x, y) = 0$  when and only when  $x$  coincides with  $y$ .
2.  $\rho(x, y) = \rho(y, x)$ .
3.  $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$  for any three elements  $x, y, z$ , belonging to  $R$  (triangle axiom).

The set  $D$  of metric space  $R$  is termed dense in the set  $D_0 \subset R$ , if for each  $x \in D_0$  and  $\varepsilon > 0$  there is a point  $z \in D$  such that  $\rho(x, z) < \varepsilon$ . It is clear that the concept of density is transitive [11], if  $D$  is dense in  $D_0$  and  $D_1$  is dense in  $D$ , then  $D_1$  is dense in  $D_0$ . Here, of course, it is assumed that the metrics are fixed.

Metric space is called separable, if it includes a denumerable dense subset.

The linear set  $R$  is called a normalized space if to each element  $x \in R$  a real number  $\|x\| \geq 0$  is associated; this number is called the norm of the elements  $x$ , and the following conditions are satisfied:

1.  $\|x\| = 0$  when and only when  $x = 0$ .
2.  $\|\lambda x\| = |\lambda| \|x\|$ .
3.  $\|x + y\| \leq \|x\| + \|y\|$ .

The sequence  $\{x_n\}$  of points of metric space  $R$  is called self-convergent if  $\rho(x_m, x_n) \rightarrow 0$  when  $m, n \rightarrow \infty$ , i.e.,  $\rho(x_m, x_n) < \varepsilon$  when  $m, n \geq N_\varepsilon$ .

The metric space  $R$  is called complete if each self-convergent sequence  $\{x_n\}$  converges, i.e., a point  $x_0 \in R$  exists such that  $x_n \rightarrow x_0$ .

The normalized space  $R$  is called unitary if in it we can introduce a scalar product associated with the norm by the relationship

$$\|x\| = \sqrt{(x, x)}.$$

The complete unitary space is called a Hilbert space<sup>(1)</sup>.

For interpolation of the spaces  $L^p$  we must have the concept of a measurable function and a measurable set. Here we shall assume that we know the concept of the outer measure [8] of the set.

The set  $R$  is called measurable if it can be closed by an open set  $D$  such that the outer measure of the difference  $R - D$  is as small as desired.

The function  $f(p)$  given on the measurable set  $R$  is called measurable if for any real  $a$  the sets  $D[f \geq a]$ ;  $D[f < a]$ ;  $D[f > a]$ ;  $D[f \leq a]$  are measurable. The symbol  $D[f \geq a]$  denotes a set of those points  $R$  for which the condition contained in the brackets is satisfied.

By the space  $L^p(s)$  we mean a set of all measurable functions given on the measurable set  $S$ , the  $R^{\text{th}}$  power of the modulus of which is integrable in the Lebesgue sense, i.e., if  $f \in L^p$ , then  $|f|^p \in L(s)$ . The norm in the space  $L^p(s)$  is interpolated from the formula

$$N_p(f) = \|f\|_p = \left( \int |f|^p dx \right)^{\frac{1}{p}}.$$

---

(1) Sometimes [5] the following definitions are used: the linear space  $R$  is called a Hilbert space if it is separable and if a scalar product is introduced in it. This definition is not equivalent to that given above. In [7] and [10] examples of nonseparable Hilbert spaces are cited.

It satisfies the following inequality:

$$N_1(f, g) \leq N_p(f) N_{p'}(g),$$

where  $p$  and  $p'$  are adjoint indices:  $1/p + 1/p' = 1$  is the Hölder inequality [8] (when  $p = 2$  it is the Buniakowski-Schwartz inequality);

$$N_p(f+g) \leq N_p(f) + N_p(g)$$

is the Minkowski inequality [8].

When  $p \rightarrow \infty$

$$N_p(f) \rightarrow \text{Max } |f|,$$

where  $\text{Max } |f|$  denotes the intrinsic upper bound [11] of  $|f|$ , i.e., the least value of  $\eta$  such that  $|f| \leq \eta$  almost everywhere. Therefore,  $L^\infty$  is denoted as a class of intrinsically bounded functions or functions of equivalent<sup>(2)</sup> bounded functions. Let us note that in the space  $C$  of all continuous functions of the norm  $N_C(f) = \text{Max } |f|$ ,  $N_C(f) = \text{Max } (f)$  is also defined but here  $\text{Max}$  is the ordinary maximum.

The set of all possible elements such as  $\sum_{i=1}^n \lambda_i \varphi_i$ , where  $\lambda_i$  are arbitrary real numbers, is called the span of the system  $\{\phi_n\}$ .

When  $n = 1, 2, \dots$ , let  $\{\phi_n(s)\}$  represent a system of nonzero functions of  $L^2(s)$ . If

$$(\varphi_m, \varphi_n) = \iint_s \varphi_m \varphi_n ds = 0$$

---

<sup>(2)</sup> The functions  $f$  and  $g$  are called equivalent if  $f = g$  almost everywhere.

when  $m \neq n$ , then we say that  $\{\phi_n\}$  is an orthogonal system on  $s$ . If furthermore,

$$(\varphi_n, \varphi_n) = \iint_s |\varphi_n|^2 ds = (\|\varphi_n\|_2)^2 = 1$$

for all  $n$ , then we say that  $\{\phi_n\}$  forms an orthogonal normalized (orthonormalized) system on  $S$ .

The system  $\{\phi_n(s)\}$  is called [11] complete in  $L^p(S)$ , where  $1 \leq p \leq \infty$ , or in  $C$  if no nonzero function exists from  $L^p(s)$  or  $C(s)$  that is orthogonal to each  $\phi_n$ , i.e., if for  $f \in L^p(s)$  ( $f \in C(s)$ ),

$$\iint_s f \varphi_n ds = 0 \quad (n=1, 2, \dots)$$

we imply  $f \equiv 0$ . Since, if  $f \in L^q(s)$ , then  $f \in L^p(s)^{(3)}$ , where  $b > d$ ; then from /19 the definition of completeness it follows that if  $\{\phi_n\}$  is complete in  $L^p(s)$  then it is complete in  $C$  and  $L^q(s)$  when  $q < p$ .

The system  $\{\phi_n\}$  of functions from  $L^p(s)$  (or  $C$ ) is called [11] closed in  $L^p(s)$  (or  $C$ ) if the span of the system  $\{\phi_n\}$  is dense in  $L^p(s)$  (or  $C$ ).

The following statement will be used often and therefore, we formulate it as a theorem.

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(3) For proof of this assumption consider the integral

$$\iint_s |f|^p ds = \iint_{s_1(|f| \leq 1)} |f|^p ds + \iint_{s_2(|f| > 1)} |f|^p ds < |s| + A,$$

where  $|s|$  is the size of the set  $s$ , and

$$A = \iint_s |f|^q ds < \infty.$$

Theorem 3.1. An orthonormalized system  $\varphi_i(M)$ , obtained from a complete (closed) system, is complete (closed).

Let us prove the following theorem [11].

Theorem 3.2. If  $\Phi_n = \sum_{m=0}^n \gamma_m \varphi_m$  is a certain polynomial of the orthonormalized system  $\{\varphi_n\}$ , then

$$N_2^2(f - \Phi_n) = N_2^2(f) - \sum_{m=0}^n C_m^2 + \sum_{m=0}^n (C_m - \gamma_m)^2,$$

where  $C_m$  are Fourier coefficients of the function  $f$  for the system  $\{\varphi_n\}$ .  
Considering the easy-to-prove equations

$$\iint_s f \Phi_n ds = \sum_{m=0}^n C_m \gamma_m, \quad \iint_s \Phi_n^2 dx = \sum_{m=0}^n \gamma_m^2,$$

we obtain

$$N_2^2(f - \Phi_n) = N_2^2(f) - 2 \sum_{m=0}^n C_m \gamma_m + \sum_{m=0}^n \gamma_m^2 = N_2^2(f) - \sum_{m=0}^n C_m^2 + \sum_{m=0}^n (C_m - \gamma_m)^2.$$

The following theorems are a direct consequence of the proved theorem.

Theorem 3.3. Of all the polynomials  $\Phi_n$  of a given order the best mean square approximation of the function is given by the Fourier polynomial

$f_n = \sum_{m=0}^n C_m \varphi_m$  and the following expressions are valid

$$N_2^2(f - f_n) = N_2^2(f) - \sum_{m=0}^n C_m^2,$$

$$\sum_{m=0}^{\infty} C_m^2 < N_2^2(f) = \iint_s f^2 ds.$$

The following important theorem, which is given without proof, will be used below.

Theorem 3.4 (Ritz-Fisher) [11]. Let  $\sum_{n=0}^{\infty} C_n^2 < \infty$ . Then a function  $f$  exists from  $L^2(s)$  which has its own Fourier coefficients  $C_n$ . Furthermore,  $f_n \rightarrow f$  in the sense of the metric  $L^2(s)$ , i.e., /20

$$\iint_s (f_n - f)^2 ds \rightarrow 0 \quad (3.1)$$

and

$$\iint_s f^2 ds = \sum_{n=0}^{\infty} C_n^2. \quad (3.2)$$

Theorem 3.5 is a direct corollary of the Ritz-Fisher theorem.

Theorem 3.5. If the orthonormalized system  $\{\varphi_n\}$  is complete,  $f \in L^2(s)$  and  $C_n$  ( $n = 1, 2, \dots$ ) is a Fourier coefficient of the function  $f$  for the system  $\{\varphi_n\}$ , then  $f$  and  $C_n$  satisfy (3.1) and (3.2).

In fact on the strength of Theorem 3.3  $\sum_{n=0}^{\infty} C_n^2 < \infty$  and  $f$  is equivalent to the function  $f$  in the Ritz-Fisher theorem.

Let us prove that closure and completeness of equivalent in  $L^2(s)$ .

Theorem 3.6. The system of functions from  $L^2(s)$  is closed when and only when it is complete.

On the strength of Theorem 3.1, it is sufficient to prove equivalence for the system  $\{\varphi_n\}$ , obtained after orthonormalization.

Let  $\{\varphi_n\}$  be complete and  $f \in L^2$ , then according to Theorem 3.5,  $f_n \rightarrow f$  in the sense of the metric  $L^2(s)$ . Hence, on the strength of the definition, the closure follows.

Let  $\{\varphi_n\}$  be closed and all Fourier coefficients of the function be equal to zero. We must prove that  $f \equiv 0$  (completeness). Since  $\{\varphi_n\}$  is closed, then a sequence of polynomials  $\Phi_n$  exists such that  $\Phi_n \rightarrow f$  in the sense of the metric

$L^2(s)$ , i.e.,  $N_2(f - \Phi_n) \rightarrow 0$ , but on the strength of Theorem 3.3,  $N_2(f - f_n) = 0$ . But  $f_n = 0$  and therefore,  $N_2(f) = 0$ ,  $f \equiv 0$  and the completeness of the system  $\{\varphi_n\}$  is proved.

Let us give without proof analogous theorems for the spaces  $L^p$  which will be used below.

Theorem 3.7. [11]. If  $1 < p < \infty$  and  $\{\varphi_n\}$  is closed in  $L^p(s)$ , then it is complete in  $L^{p'}$ .

Theorem 3.8 [11]. If  $1 < p < \infty$  and  $\{\varphi_n\}$  is complete in  $L^p(s)$ , then it is closed in  $L^{p'}$ .

When  $p = 1$  Theorem 3.8 is not valid.

Below we shall use the following Theorem [11].

Theorem 3.9. If  $1 < p < \infty$ , then the sets of functions of the space  $L^q(s)$  ( $p < q < \infty$ ), and also the sets of bounded  $B(s)$ , of continuous  $C(s)$  which have a continuous  $k^{\text{th}}$  derivative of the  $C_k$  function are complete in  $L^p(s)$ .

Using Theorem 3.9, we can prove the following theorem.

/21

Theorem 3.10. If the system  $\{\varphi_i(M)\}$  is closed in  $L^p(s)$  and  $1 < q < p$ , then the system  $\{\varphi_i(M)\}$  is also closed in  $L^q(s)$ . For proof let us look at the mean value of the function  $f$  in the interval  $(a, b)$  with the index  $p$  [11].

$$M_p(f) = \left( \frac{1}{b-a} \int_a^b |f|^p dx \right)^{\frac{1}{p}}.$$

The mean values have the property [11]

$$M_q(f) < M_p(f) \text{ when } q < p.$$

Let us note that the norm  $N_p(f)$  does not have this property. Let  $f \in L^q(s)$ ; then according to Theorem 3.9 we find such a function  $g \in L^p(s)$  that  $N_q(f-g) < \frac{1}{2} \varepsilon$ . But  $\{\varphi_i(M)\}$  is closed in  $L^p(s)$  and therefore, there exists such a polynomial  $\psi$ , that

$$N_p(g-w) < \frac{1}{2} \varepsilon (b-a)^{\frac{1}{p}-\frac{1}{q}}.$$

If we take into account the trivial equation

$$N_p(f) = (b-a)^{\frac{1}{p}} M_p(f)$$

and the mean value property, we find

$$N_q(f) \leq (b-a)^{\frac{1}{q}-\frac{1}{p}} N_p(f).$$

Therefore, considering Minkowski's inequality, we have

$$N_q(f-w) \leq N_q(f-g) + N_q(g-w) < \frac{1}{2} \varepsilon + (b-a)^{\frac{1}{q}-\frac{1}{p}} N_p(g-w) < \varepsilon.$$

Let us note that if  $\{\varphi_n\}$  is closed in  $C(s_1)$ , then it also is closed in  $L^p(s)$  when  $1 < p < \infty$  (but not necessarily closed in  $L^\infty$ ).

Let  $G$  be a certain subspace of the Hilbert space  $R$ , formed by the system  $\{\varphi_i\}$  ( $G$  is the span of the system  $\{\varphi_i\}$ ). We can prove the following theorem.

Theorem 3.11 [10]. If there exists in  $G$  an element (vector)  $y$ , which is the least distance from  $x \in R$  ( $x$  does not belong to  $G$ ), then the vector  $x-y$  is orthogonal to each vector  $g$  from  $G$ , i.e.,

$$(x-y, g) = 0 \quad (g \in G)$$

and in the case of a finite-dimensional subspace G, formed by the linearly independent vectors  $\varphi_1, \varphi_2, \dots, \varphi_n$  for the square of the error with which the vector y approximates the vector x, we have

$$\delta^2 = \min_{\alpha_k} \|\bar{x} - \alpha_1 \varphi_1 - \alpha_2 \varphi_2 - \dots - \alpha_n \varphi_n\|^2 = \frac{G(x, \varphi_1, \varphi_2, \dots, \varphi_n)}{G(\varphi_1, \varphi_2, \dots, \varphi_n)}, \quad (3.3)$$

where

/22

$$y = \sum_{i=1}^n \alpha_i \varphi_i,$$

$$G(\varphi_1, \varphi_2, \dots, \varphi_n) = \begin{vmatrix} (\varphi_1, \varphi_1) & (\varphi_2, \varphi_1) & \dots & (\varphi_n, \varphi_1) \\ (\varphi_1, \varphi_2) & (\varphi_2, \varphi_2) & \dots & (\varphi_n, \varphi_2) \\ \dots & \dots & \dots & \dots \\ (\varphi_1, \varphi_n) & (\varphi_2, \varphi_n) & \dots & (\varphi_n, \varphi_n) \end{vmatrix}$$

is the Gram determinant of the system of vectors  $\varphi_1, \varphi_2, \dots, \varphi_n$ .

Let us assume that a vector f exists in G for which

$$(x - y, f) = r \neq 0,$$

and analyze the vector

$$z = y + \frac{r}{(f, f)} f \in G.$$

For z we have

$$\|x - z\|^2 = (x - y - \frac{r}{(f, f)} f, x - y - \frac{r}{(f, f)} f) = \|x - y\|^2 - \frac{|r|^2}{(f, f)} < \|x - y\|^2$$

and the resultant contradiction (on the assumption that y is a near point to x) proves the first part of the theorem. For proof of (3.3) we write in detail the equations

$$(x - y, \varphi_k) = 0, \quad k = (1, 2, \dots, n), \quad (3.4)$$

using the expression for y,

$$\sum_{i=1}^n \alpha_i (\varphi_i, \varphi_k) = (x, \varphi_k), \quad k = (1, 2, \dots, n). \quad (3.5)$$

Taking (3.4) into account, for  $\delta^2$  we find

$$\delta^2 = (x-y, x-y) = (x-y, x) - (x-y, y) = (x-y, x) = (x, x) - (y, x),$$

or expanding the expression  $(y, x)$

$$(x, x) - \delta^2 = \sum_{i=1}^n \alpha_i (\varphi_i, x).$$

Combining (3.6) with System (3.5) we find a system of  $n + 1$  equations with  $n$  unknowns  $(\alpha_1, \alpha_2, \dots, \alpha_n)$ .

To solve this system it is necessary and sufficient that the rank of its matrix be equal to the rank of the resolved matrix

$$A = \begin{vmatrix} (\varphi_1, \varphi_1) & \dots & (\varphi_n, \varphi_1) & (x, \varphi_1) \\ \vdots & \ddots & \vdots & \vdots \\ (\varphi_1, \varphi_n) & \dots & (\varphi_n, \varphi_n) & (x, \varphi_n) \\ (\varphi_1, x) & \dots & (\varphi_n, x) & (x, x) - \delta^2 \end{vmatrix}.$$

Hence we find that  $A = 0$  and

/23

$$\delta^2 = \frac{G(x, \varphi_1, \varphi_2, \dots, \varphi_n)}{G(\varphi_1, \varphi_2, \dots, \varphi_n)}. \quad (3.7)$$

We can prove that

$$G(\varphi_1, \varphi_2, \dots, \varphi_n) < G(\varphi_1, \varphi_2, \dots, \varphi_m) G(\varphi_{m+1}, \varphi_{m+2}, \dots, \varphi_n), \quad (3.8)$$

where  $m < n$  and  $\varphi_1, \varphi_2, \dots, \varphi_n$  are linearly independent vectors.

It is obvious that

$$\min_{\alpha} \| \varphi_k - \alpha_{k+1} \varphi_{k+1} - \dots - \alpha_n \varphi_n \| \leq \min_{\beta} \| \varphi_k - \beta_{k+1} \varphi_{k+1} - \dots - \beta_m \varphi_m \|$$

and

$$\min_{\alpha} \| \varphi_m - \alpha_{m+1} \varphi_{m+1} - \dots - \alpha_n \varphi_n \| \leq \| \varphi_m \|.$$

Taking into account these latter inequalities, from (3.7) we find

$$\begin{aligned} \frac{G(\varphi_k, \varphi_{k+1}, \dots, \varphi_n)}{G(\varphi_{k+1}, \dots, \varphi_n)} &\leq \frac{G(\varphi_k, \varphi_{k+1}, \dots, \varphi_m)}{G(\varphi_{k+1}, \dots, \varphi_m)}, \\ \frac{G(\varphi_m, \varphi_{m+1}, \dots, \varphi_n)}{G(\varphi_{m+1}, \dots, \varphi_n)} &\leq G(\varphi_m). \end{aligned} \quad (3.9)$$

Since  $G(\varphi_j, \varphi_{j+1}, \dots, \varphi_i) > 0$  ( $j < i$ ), then (3.9) can be written in the form

$$\frac{G(\varphi_k, \varphi_{k+1}, \dots, \varphi_n)}{G(\varphi_k, \varphi_{k+1}, \dots, \varphi_m)} \leq \frac{G(\varphi_{k+1}, \varphi_{k+2}, \dots, \varphi_n)}{G(\varphi_{k+1}, \varphi_{k+2}, \dots, \varphi_m)} \quad (k=1, 2, \dots, m-1).$$

Thus,

$$\frac{G(\varphi_1, \dots, \varphi_n)}{G(\varphi_1, \dots, \varphi_m)} \leq \frac{G(\varphi_2, \dots, \varphi_n)}{G(\varphi_2, \dots, \varphi_m)} \leq \dots \leq \frac{G(\varphi_m, \dots, \varphi_n)}{G(\varphi_m)} \leq G(\varphi_{m+1}, \dots, \varphi_n).$$

#### §4. Linear Independence and Completeness of Several Systems of Harmonic Functions.

Proof of convergence of the approximate method discussed below for solving the Problem (1.1) - (1.2) is based on the linear independence and completeness of a certain system of harmonic functions. Therefore, we shall give several proofs for completeness of this system.

Let  $G_1$  be a region with a sufficiently smooth boundary  $S$  (it is sufficient, for example, that  $S_1$  be a Lyapunov surface<sup>(4)</sup>) which completely includes the

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(4) All the material discussed below is valid also for the two-dimensional region. In the latter case only the System [1-3] of functions  $\{\ln r(M_i, M)\}$  which differs from (4.1) is analyzed.

region  $G$  and the minimal distance from  $S$  to  $S_1$  be greater than zero, i.e., the surface  $S_1$  is not tangent to the surface  $S$ .

Let us introduce the definition:

$$\frac{1}{r(M_i, M)} = \omega_i(M) \quad (i=1, 2, \dots), \quad (4.1)$$

where  $M_i \in S_1$  are elements of the denumerable set of points which are everywhere dense on the surface  $S_1$ .

Theorem 4.1 The system of functions  $\{\omega_i(M)\}$  is linearly independent on the curve  $S$ , i.e., for any  $N$  from the equation /24

$$\sum_{i=1}^N C_i \omega_{k_i}(M) \equiv 0, \quad M \in S \quad (4.2)$$

it follows that

$$\sum_{i=1}^N |C_i| = 0,$$

where all  $k_i$  are whole numbers.

Let us assume the opposite: let there be found such bounded number  $C_i$  which are not all equal to zero that for a certain  $N$  Expression (4.2) is satisfied and a certain  $C_r \neq 0$  ( $r \leq N$ ). From (4.2) and the theorem of uniqueness for the Dirichlet problem it follows that

$$\sum_{i=1}^N C_i \omega_{k_i}(M) \equiv 0, \quad M \in \bar{G} = G + S, \quad (4.3)$$

and from the analyticity of the left-hand side of (4.3) it follows that

$$\sum_{i=1}^N C_i \omega_{k_i}(M) \equiv 0, \quad M \in G_1. \quad (4.4)$$

Let  $M$  approach  $M_{k_r}$ . Then  $|C_r \omega_{k_r}| \rightarrow \infty$ , and all other terms in (4.4) remain bounded, thus contradicting Equation (4.4) and consequently our assumption that  $C_r \neq 0$ .

From Theorem 4.1 there follows the validity of the following theorem.

Theorem 4.2. We can construct such a system of functions  $\{\varphi_i(M)\}$  orthonormalized on  $S$  such that

$$\varphi_i(M) = \sum_{k=1}^i A_{i,k} \omega_k(M), \quad (4.5)$$

where  $A_{i,k}$  are the coefficients of orthonormalization.

The system  $\{\varphi_i(M)\}$  can be constructed [5] successively. Since the system  $\{\omega_i(M)\}$  is linearly independent, then

$$\iint_S \varphi_i^2 ds > 0 \quad (i=1, 2, \dots).$$

Therefore, we have

$$\varphi_1(M) = \frac{\omega_1(M)}{\sqrt{\iint_S \omega_1^2 ds}}.$$

It is clear that  $\varphi_1(M)$  is normalized. Let us construct the elements  $\bar{\varphi}_2(M) = \omega_2(M) + \alpha_{2,1} \varphi_1$  orthogonal to  $\varphi_1$

$$\iint_S \varphi_1 \bar{\varphi}_2 ds = \iint_S \varphi_1 \omega_2 ds + \alpha = 0.$$

/25

For  $\alpha_{2,1}$  we find the equation

$$\alpha_{2,1} = - \iint_S \varphi_1 \omega_2 ds.$$

From the linear independence of the system  $\{\omega_i(M)\}$  it directly follows that

$$\iint_S \bar{\varphi}_2^2 ds > 0.$$

Therefore, we have

$$\varphi_2 = \frac{\bar{\varphi}_2}{\sqrt{\iint_S \bar{\varphi}_2^2 ds}},$$

which guarantees both normalization of the element  $\varphi_2$  and orthogonality of  $\varphi_1$  and  $\varphi_2$ . Let us construct the orthonormalized elements  $\varphi_1, \varphi_2, \dots, \varphi_k$ . The next element will be sought in the form

$$\varphi_{k+1} = \frac{\bar{\varphi}_{k+1}}{\sqrt{\iint_S \bar{\varphi}_{k+1}^2 ds}}, \quad (5.6)$$

where

$$\bar{\varphi}_{k+1} = \omega_{k+1} + \alpha_{k+1,1} \varphi_1 + \dots + \alpha_{k+1,k} \varphi_k. \quad (4.7)$$

The coefficients  $\alpha_{k+1,i}$  ( $i=1,2,\dots,k$ ) are determined from the condition of orthogonality  $\iint_S \bar{\varphi}_{k+1} \varphi_i ds = 0$  ( $i=1,2,\dots,k$ ). We find

$$\alpha_{k+1,i} = \iint_S \omega_{k+1} \varphi_i ds.$$

From Expression (4.7), if we take into account that  $\varphi_1, \varphi_2, \dots, \varphi_k$  do not contain  $\omega_{k+1}$ , it follows that the denominator of the right-hand side of (4.6) is nonzero. Thus, we construct any element of the orthonormalized system  $\{\varphi_i(M)\}$ . To obtain Expression (4.5) we must substitute the values  $\varphi_1, \varphi_2, \dots, \varphi_k$  into (4.6) expressed through  $\omega_1, \omega_2, \dots, \omega_k$ . Theorem 4.2 is proved.

The algorithm described above gives  $\varphi_i(M)$  in the following form:

$$\varphi_i(M) = \sum_{k=1}^{i-1} \bar{A}_{i,k} \varphi_k + \bar{A}_{i,i} \omega_{i+1}. \quad (4.8)$$

As will be shown in §5, of this Chapter [for the discussed approximate method  $\varphi_i(M)$ ] we must bear in mind (4.5) because  $A_{i,k}$  directly participates in the algorithm of the solution. From (4.8), we can obtain the coefficients

$A_{ik}$  of Expression (4.5). In fact, it is easy to prove the following expression:

$$A_{t,i} = \bar{A}_{t,i}, \quad A_{t,k} = \sum_{j=k}^{i-1} \bar{A}_{t,j}. \quad (4.9)$$

Since in any computer center there are standard subprograms for computing the /26  
integral with any preassigned accuracy and for computing the determinant, then  
from the viewpoint of simplicity of carrying out machine computation, the follow-  
ing algorithm for computing the coefficients  $A_{i,k}$  possesses a certain advantage  
over that described above.  $A_{i,k}$  are computed from the following expression [6].

$$\varphi_n = \frac{\begin{vmatrix} \iint \omega_1^2 ds, & \iint \omega_1 \omega_2 ds, & \dots, & \iint \omega_1 \omega_{n-1} ds, & \omega_1 \\ \iint \omega_2 \omega_1 ds, & \iint \omega_2^2 ds, & \dots, & \iint \omega_2 \omega_{n-1} ds, & \omega_2 \\ . & . & . & . & . \\ . & . & . & . & . \\ \iint \omega_n \omega_1 ds, & \iint \omega_n \omega_2 ds, & \dots, & \iint \omega_n \omega_{n-1} ds, & \omega_n \end{vmatrix}}{\sqrt{G_{n-1} \cdot G_n}}, \quad (4.10)$$

where  $G_n$  is the Gram determinant of the functions  $\omega_1, \omega_2, \dots, \omega_n$

$$G_n = \begin{vmatrix} \iint \omega_1^2 ds, & \iint \omega_1 \omega_2 ds, & \dots, & \iint \omega_1 \omega_n ds \\ \iint \omega_1 \omega_2 ds, & \iint \omega_2^2 ds, & \dots, & \iint \omega_2 \omega_n ds \\ \dots & \dots & \dots & \dots \\ \iint \omega_n \omega_1 ds, & \iint \omega_n \omega_2 ds, & \dots, & \iint \omega_n^2 ds \end{vmatrix}. \quad (4.11)$$

The integrals in (4.10) and (4.11) are taken on the surface  $S$ .

Let us prove the following assumption.

Theorem 4.3. The Gram Determinant (4.11) for the linearly independent system  $\{\omega_i(M)\}$  is nonzero.

Let us assume the opposite. Let  $G_n = 0$ . We can analyze the system of linear equations relative to  $\alpha(\alpha_1, \alpha_2, \dots, \alpha_n)$  written in vector form

$$G_n \alpha = 0. \quad (4.12)$$

Since the determinant of this system is equal to zero, then there exists a nontrivial solution  $\alpha(\alpha_1, \alpha_2, \dots, \alpha_n)$ .

Let us show that

$$\delta = \iint_s \left( \sum_{i=1}^n \alpha_i \omega_i \right)^2 ds = 0. \quad (4.13)$$

In fact

$$\delta = \sum_{j=1}^n \alpha_j \sum_{i=1}^n \alpha_i \iint_s \omega_i \omega_j ds.$$

But since  $\alpha_1, \alpha_2, \dots, \alpha_n$  are nontrivial solutions to System (4.13), it is then /27 clear that

$$\sum_{i=1}^n \alpha_i \iint_s \omega_i \omega_j ds = 0 \quad (j=1, 2, \dots, n)$$

and therefore  $\delta = 0$  or

$$\sum_{i=1}^n \alpha_i \omega_i = 0.$$

The resultant contradiction (linear dependence of the system  $\{\omega_i(M)\}$ ) proves Theorem 4.3.

However, we must note that with the approximate method (with a finite number of digits) for accomplishing the above algorithms the first approach possesses a substantial advantage. This problem will be analyzed below.

Let us prove the following theorem.

Theorem 4.4. The orthonormalized system of functions  $\{\varphi_i(M)\}$  is closed in the space  $L^2(s)$  of the square-integrable functions given on the boundary  $S$ .

First let us prove the completeness of the system of functions  $\{\varphi_i(M)\}$  in the space  $L^2(s)$ . Let  $\gamma(M) \in L^2(s)$ . We can analyze the function

$$\iint_s \gamma(M) \frac{1}{r(M, N)} ds_M, \quad N \in S_1.$$

This function, continuous on  $S_1$ , assumes zero values on the everywhere dense set of points  $N_i \in S_1$ ; therefore

$$\iint_s \gamma(M) \frac{1}{r(M, N)} ds_M \equiv 0, \quad N \in S_1. \quad (4.14)$$

But (4.14) is the potential of a single layer and since on the closed Lyapunov surface  $S_1$  and at infinity it is equal to zero, then on the strength of the uniqueness of the solution to the external Dirichlet problem (uniform conditions at infinity) it is everywhere equal to zero in  $V_e$ . This is possible only in the event [12] if the density of the potential is equal to zero. Thus, the completeness of the system  $\{\omega_i(M)\}$  in the space  $L^2(s)$  is proven.

But on the strength of Theorem 3.6, the system  $\{\omega_i(M)\}$  is closed. For final proof of Theorem 4.4, we must use Theorem 3.1.

From the proved theorem and the definition of completeness it directly follows that the system  $\{\varphi_i(M)\}$  is complete in  $C$  and  $L^q(s)$ , where  $q > 2$ . As far as the closure is concerned, from Theorem 3.10 it follows that our system  $\{\varphi_i(M)\}$  is closed in the space  $L^q(s)$ , where  $q < 2$ . Therefore, from Theorem 4.4, it does not follow that the system  $\{\varphi_i(M)\}$  is closed in  $C$  or  $L^q(s)$  when

$q > 2$ . Now let us prove the theorem that the analyzed system is closed in  $C$  and consequently closed in  $L^q(s)$  for all  $q < \infty$  ( $q \geq 1$ ).

Theorem 4.5. The system  $\{\varphi_i(M)\}$  is closed in the space  $C(s)$  of all continuous functions, i.e., for any function  $\gamma(M) \in C(s)$  and for any  $\varepsilon > 0$  there is an  $N_0(\varepsilon)$  and a system of coefficients  $a_i$  ( $i = 1, 2, \dots, n$ ) such that if  $n > N_0(\varepsilon)$ , then

$$\max_{M \in S} \left| \gamma(M) - \sum_{i=1}^n a_i \varphi_i(M) \right| < \varepsilon.$$

For proof of this theorem we use the following statement from Reference [13]: any function that is continuous on the surface  $S$  may be uniformly approximated by means of harmonic polynomials, if the region  $V$  with the boundary  $s$  contains a stable solution of the Dirichlet problem with respect to deformation of the region<sup>(5)</sup>.

Let  $P_m(M)$  be a harmonic polynomial of order  $m$  for the function  $\gamma(M) \in C(s)$

$$\max_{M \in S} |\gamma(M) - P_m(M)| < \frac{\varepsilon}{2}.$$

Let us analyze the internal Dirichlet problem in the region  $V_1$  with the boundary  $s_1$

$$\begin{aligned} \Delta u &= 0 \quad \text{in } V_1, \\ u|_{s_1} &= P_m|_{s_1}, \end{aligned} \tag{4.15}$$

and write its solution in the form of the potential of a single layer

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(5) In the two-dimensional case when  $s$  is a curve an analogous statement directly follows from the theorems corresponding to the Weierstrass theorems in the complex region (see, for example, the Runge Theorem [3], concerning the uniform approximation of the function of the complex variable by a complex polynomial).

$$u(M) = \iint_{S_1} \Omega(M) \ln \frac{1}{r(x, M)} ds_x, \quad (4.16)$$

where the density  $\Omega(M)$  is expressed through the normal derivatives of the solutions to the internal and external Dirichlet problems. Sufficient smoothness of  $S_1$  produces smooth boundary values for Problem (4.15) and therefore, guarantees existence of these derivatives. Let us look at Expression (4.16) for the points  $M_i \in S_1$ . The integrand will be bounded and continuous (if the density is continuous) and therefore, the integral can be replaced by a Riemann sum with the number of terms  $m_1$  (or by some kind of cubature formula with nodes at the point  $x_i = M_i \in S_1$ ), such that

$$\max_{M \in S} \left| \iint_{S_1} \Omega(M) \ln \frac{1}{r(x, M)} ds_x - \sum_{i=1}^{m_1} a_i \ln \frac{1}{r(x_i, M)} \right| < \frac{\varepsilon}{2}.$$

Let  $n \geq \max(m, m_1) = N_0(\varepsilon)$ ; then we find

$$\begin{aligned} \max_{M \in S} \left| \gamma(M) - \sum_{i=1}^n a_i \omega_i(M) \right| &\leq \max_{M \in S} \left| \gamma(M) - P_m(M) \right| + \\ &+ \max_{M \in S} \left| P_m(M) - \sum_{i=1}^n a_i \omega_i(M) \right| < \frac{\varepsilon}{2} + \max_{M \in S} \left| \iint_{S_1} \Omega(M) \ln \frac{1}{r(x, M)} ds_x - \right. \\ &\left. - \sum_{i=1}^n a_i \ln \frac{1}{r(x_i, M)} \right| < \varepsilon \end{aligned} \quad /29$$

and the closure of the system  $\{\omega_i(M)\}$  is proved. To complete proof of Theorem 4.5, we must use Theorem 3.1.

## § 5. Approximate Method for Solving the Dirichlet Problems

Let us analyze the Problem (1.1) - (1.2) and, after substituting  $\Gamma = 1/4\pi \cdot 1/r$  into it, write the basic formula for the theory of harmonic functions (2.12) for the points  $x \in V$  and  $x \in V_e$ , respectively

$$u(x) = \frac{1}{4\pi} \iint_S \frac{d}{dn} \frac{1}{r(x, M)} \psi(M) ds_M - \frac{1}{4\pi} \iint_S \frac{1}{r(x, M)} \varphi(M) ds_M, \quad x \in V \quad (5.1)$$

$$0 = \frac{1}{4\pi} \iint_S \frac{d}{dn} \frac{1}{r(x, M)} \psi(M) ds_M - \frac{1}{4\pi} \iint_S \frac{1}{r(x, M)} \varphi(M) ds_M, \quad x \in V_e, \quad (5.2)$$

where

$$\psi(M) = u \Big|_S, \quad \varphi(M) = \frac{du}{dn} \Big|_S.$$

Since in the case of the Dirichlet problem the function  $\psi(M)$  is given, then (5.2) can be written in the form

$$\iint_S \frac{1}{r(x, M)} \varphi(M) ds_M = F(x), \quad (5.3)$$

where  $F(x)$  is a known function

$$F(x) = \iint_S \psi(M) \frac{d}{dn} \frac{1}{r(x, M)} ds_M.$$

Below we shall show that from Condition (5.3) we can determine the function  $\varphi(M)$  for the Dirichlet problem in the following manner. We can construct coefficients for expanding the unknown function  $\varphi(M)$  into a Fourier series for the complete system  $\{\varphi_i(M)\}$ , obtained by orthonormalization of System (4.1). After substituting the approximate values found for the function  $\varphi(M)$  into Formula (5.1) and carrying out the necessary cubatures, we find the approximate value of the solution to the Dirichlet Problem (1.1) - (1.2) at any point of the region  $V$ .

The function determined on  $S$ , satisfying (5.3) for the arbitrary point  $x$  lying outside the closed region  $V$ , will be called the solution to Equation (5.3). If we analyze the normal derivative of both sides of (5.3) by passing to the limit when  $x \rightarrow M_0 \in S$  and considering Expression (2.24), we find

$$\varphi(M_0) + \frac{1}{2\pi} \iint_S \varphi(M) \frac{d}{dn} \frac{1}{r(M_0, M)} ds_M = \lim_{x \rightarrow M_0} \frac{d}{dn_x} \left[ \frac{1}{2\pi} F(x) \right]. \quad (5.4) \quad \underline{/30}$$

For existence of the limit in the right-hand side of (5.4) it is sufficient to require the continuity of  $\psi'(M)$  (as follows from the Lyapunov example [4], satisfaction of the Hölder condition for the function  $\psi(M)$  is insufficient for existence of normal derivatives of the potential of the double layer). Equation (5.4) is an integral equation of the external Neumann problem and as we know [14], is uniquely resolvable. Let us show that the solution to Equation (5.4) also satisfies the functional Equation (5.3). For this let us substitute the solution to Equation (5.4) into (5.3) and denote by  $v(x)$  the function obtained

$$v(x) = \frac{1}{4\pi} \iint_s \frac{d}{dn} \left[ \frac{1}{r(x, M)} \right] \psi(M) ds_M - \frac{1}{4\pi} \iint_s \frac{1}{r(x, M)} \varphi(M) ds_M. \quad (5.5)$$

We must show that  $v(x) \equiv 0$  when  $x \in V_c$ . In the right-hand side of Equation (5.5) we have the sum of potentials of the single and double layer, and therefore,  $v(x)$  is a harmonic function. From (5.4) it follows that at the boundary  $s$  of the region  $V$  the function  $v(x)$  takes zero values. If, furthermore, it is shown that

$$\lim_{|x| \rightarrow 0} v(x) = 0, \quad (5.6)$$

then from the uniqueness of the solution to the external Neumann problem we find  $v(x) \equiv 0$ . For the three-dimensional case Equation (5.6) follows from the very form of the right-hand side of (5.5). As far as the two-dimensional case is concerned, the Equations (5.4) and (5.5) take, respectively, the following form:

$$\begin{aligned} & \varphi(M_0) + \frac{1}{2\pi} \int_s \varphi(M) \frac{d}{dn} \ln \frac{1}{r(M_0, M)} ds_M = \\ & = \lim_{x \rightarrow M_0} \frac{d}{dn} \left\{ \frac{1}{\pi} \int_s \psi(M) \frac{d}{dn} \ln \frac{1}{r(x, M)} ds_M \right\}, \\ & v(x) = \frac{1}{2\pi} \int_s \frac{d}{dn} \left[ \ln \frac{1}{r(x, M)} \right] \psi(M) ds_M - \frac{1}{2\pi} \int_s \ln \frac{1}{r(x, M)} \varphi(M) ds_M, \end{aligned}$$

and according to the known properties of the harmonic potentials

$$\int \varphi(M) ds_{\mu} = 0. \quad (5.7)$$

Hence, as we know [4], we must prove the property for  $v(x)$  also in the two-dimensional case, since Condition (5.7) is sufficient for equating the second integral (potential of the simple layer) to zero in the expression for  $v(x)$ . We must show that the functional Equation (5.3) has a unique solution. For this let us look at the homogeneous functional equation

$$\iint_s \varphi(M) \frac{1}{r(x, M)} ds = 0, \quad x \in V_e, \quad (5.8)$$

and show that it has only a trivial solution. Differentiating with respect to the normal (5.8) passing to the limit when  $x \rightarrow M_0 \in s$  and taking into account the /31 expressions in (2.24), we find

$$\varphi(M_0) + \frac{1}{2\pi} \iint_s \varphi(M) \frac{1}{r(M_0, M)} ds_{\mu} = 0.$$

This equation for the external Neumann problem with homogeneous boundary values, as we know [14], also has only a zero solution.

Let us now proceed to writing the algorithm of the approximate method for solving the Dirichlet boundary value problem (1.1) - (1.2).

By  $\Phi_i$  let us denote the Fourier coefficients for expanding the function  $\varphi(M)$  in a series of functions  $\varphi_i(M)$

$$\Phi_i = \iint_s (\varphi(M)) \varphi_i(M) ds,$$

where  $\varphi_i(M)$  is determined from (4.5)

$$\varphi_i(M) = \sum_{k=1}^i A_{i,k} \omega_k(M).$$

Let us write (5.3) for the points  $M_k \in S_1$  in the form

$$\iint_S \varphi(M) \omega_k(M) ds = F_k, \quad (5.9)$$

where

$$F_k = \iint_S \psi(M) \frac{d}{dn} \omega_k(M) ds,$$

and the values of  $\omega_k$  are determined from (4.1).

Multiplying the first  $i$  in the equations by the coefficients  $A_{k,i}$  ( $k = 1, 2, \dots, i$ ) and combining we find

$$\iint_S \varphi(M) \sum_{k=1}^i A_{k,i} \omega_k(M) ds = \iint_S \varphi(M) \varphi_i(M) ds = \Phi_i = \sum_{k=1}^i A_{k,i} F_k.$$

Since in the case of the Dirichlet problem the values of  $F_k$  are unknown, and those of  $A_{k,i}$  are found in the process of orthonormalizing the system  $\{\omega_i(M)\}$ , then the Fourier coefficients of the unknown function  $\varphi(M)$  are computed. Let us introduce the symbols,

$$\varphi^{(N)}(M) = \sum_{i=1}^N \Phi_i \varphi_i(M),$$

$$u^{(N)}(x) = \frac{1}{4\pi} \iint_S \frac{1}{r(x, M)} \varphi^{(N)}(M) ds_M + \frac{1}{4\pi} \iint_S \frac{d}{dn} \left[ \frac{1}{r(x, M)} \right] \psi(M) ds_M, \quad x \in V.$$

From Theorem 3.5 there directly follows the following asymptotic equation:

$$\lim_{N \rightarrow \infty} \left\| \varphi(M) - \sum_{i=1}^N \Phi_i \varphi_i(M) \right\|_{L_2} = 0. \quad (5.10)$$

Futhermore, it can be easily proved that for any interior point  $x$  of the region  $V$  and for any  $\varepsilon > 0$  such a value of  $N_0$  can be found that if  $N_0 < N$ , then /32

$|u(x) - u^{(N)}(x)| < \varepsilon$ . In fact it is clear that

$$\begin{aligned} |u(x) - u^{(N)}(x)| &= \frac{1}{4\pi} \left| \iint_S \frac{1}{r(x, M)} \left[ \varphi(M) - \varphi^{(N)}(M) \right] ds \right| < \\ &< \frac{1}{4\pi} \iint_S \left| \frac{1}{r(x, M)} \left[ \varphi(M) - \varphi^{(N)}(M) \right] \right| ds. \end{aligned} \quad (5.11)$$

On the strength of Expression (5.10) we can select  $N$  such that the following inequality will be satisfied

$$\left\{ \iint_S [\varphi(M) - \varphi^{(N)}(M)]^2 ds \right\}^{1/2} < \frac{4\pi\epsilon\sigma}{V|S|}, \quad (5.12)$$

where  $\sigma$  is the minimal distance from point  $x$  to the boundary  $S$ ,  $|S|$  is the area of the surface  $S$ . Substituting (5.12) into (5.11) and using the Buniakowski-Schwarz inequality, we obtain

$$|u(x) - u^{(N)}(x)| < \frac{1}{4\pi} \left\{ \iint_S \left[ \frac{1}{r(x, M)} \right]^2 ds \right\}^{1/2} \left\{ \iint_S [\varphi(M) - \varphi^{(N)}(M)]^2 ds \right\}^{1/2} < \epsilon.$$

This method for approximate solution to the boundary value problems will be termed the method of V.D. Kupradze or the method of functional equations.

The completeness of the system of functions  $\{\varphi_i(M)\}$  makes it possible to use the following algorithm for approximate solution of the Dirichlet Problem (1.1) - (1.2). For a given function  $\psi(M)$  [the boundary condition of the unknown harmonic function  $u(x)$ ], let us construct a Fourier series on the basis of  $\{\varphi_i(M)\}$ :

$$\psi(M) = \sum_{k=1}^N f_k \varphi_k(M), \quad f_k = \iint_S \psi(M) \varphi_k(M) ds.$$

Then on the strength of the completeness of the system  $\{\varphi_i(M)\}$  in the sense of the metric of the space  $L_2$ , we have

$$\lim_{N \rightarrow \infty} \iint_S \left[ \psi(M) - \sum_{k=1}^N f_k \varphi_k(M) \right]^2 ds = 0.$$

The series

$$u(x) = \sum_{k=1}^{\infty} f_k \varphi_k(x)$$

for any  $x \in V$  converges and represents a solution to the Problem (1.1) - (1.2). In fact let  $G(x, M)$  be a Green function of the Dirichlet problem for the region  $V$ . Then from the existence theorem it follows that the solution may be represented in the form

$$u(x) = \iint_s \psi(M) \frac{\partial G}{\partial n} ds.$$

Let us introduce the notation

/33

$$u^{(N)}(x) = \sum_{k=1}^N f_k \varphi_k(M), \quad x \in V,$$

and look at the difference  $|u(x) - u^{(N)}(x)|$ . Using the Buniakowski-Schwarz inequality and considering the finiteness of the integral

$$\iint_s \left[ \frac{\partial G}{\partial n} \right]^2 ds$$

when  $x \in V$ , we find just as above

$$u(x) = \lim_{N \rightarrow \infty} u^{(N)}(x).$$

Let us mention that unlike this method, the method of V.D. Kupradze permits finding a solution to the Dirichlet problem with the aid of the boundary values of the normal derivative of the unknown function which are obtained as a solution to the functional Equation (5.3). In this connection we must bear in mind that in practice problems are often encountered in which it is of interest to find namely the boundary values of the normal derivative. This also explains the compilation of special tables [15] for computing the boundary values of the normal derivative. For such problems the method of V.D. Kupradze has the advantage over the above-described method.

Let us make several comments [22-24] concerning use of the discussed approximate method for solving the boundary value problems.

Following [16] the system of linearly independent functions will be termed reliable if

$$\lim_{n \rightarrow \infty} G_n = G > 0 \quad (G_n = G(\varphi_1, \varphi_2, \dots, \varphi_n))$$

When  $G = 0$  the system will be termed unreliable. Let us prove the theorem.

Theorem 5.1. The system of functions  $\{\omega_i(M)\}$  is unreliable, i.e., for any  $\varepsilon > 0$  we find such a value for  $N$  that for any  $n > N$  the following inequality will be satisfied,

$$G_n < \varepsilon.$$

We shall assume that the points  $x_i$  ( $i = 1, 2, \dots, n$ ) are renumbered in such a way that  $r(x_i, x_{i+1}) < h$ , where  $h \rightarrow 0$  when  $n \rightarrow \infty$ . It is clear that on the strength of the everywhere dense distribution of points  $x_i$  this is always possible. From (3.8) we find for an even  $n = 2k$

$$G_n < \prod_{i=1}^k G_2(\omega_{2i-1}, \omega_{2i}),$$

and with odd  $n = 2k + 1$ .

$$G_n < \int_s \omega_n^2 ds \prod_{i=1}^k G_2(\omega_{2i-1}, \omega_{2i}). \quad /34$$

For  $G_2(\omega_{2i-1}, \omega_{2i})$  we obtain

$$G_2(\omega_{2i-1}, \omega_{2i}) = \iint_s \omega_{2i-1}^2 ds \iint_s \omega_{2i}^2 ds - \left( \iint_s \omega_{2i-1} \omega_{2i} ds \right)^2.$$

Taking into account that

$$\omega_{2i-1} = \frac{1}{r(x_{2i-1}, M)} = \frac{1}{r(x_{2i}, M) + \xi(M)}, \quad (5.13)$$

where

$$|\xi(M)| = |r(x_{2i-1}, M) - r(x_{2i}, M)| \leq r(x_{2i-1}, x_{2i}) < h,$$

for  $G_2(\omega_{2i-1}, \omega_{2i})$  we find with an accuracy up to terms of higher order of smallness with respect to  $h$

$$G_2(\omega_{2i-1}, \omega_{2i}) \approx O(h) \quad (5.14)$$

and for  $G_n$  we will have

$$G_n \approx O(h^k).$$

Since  $h \rightarrow 0$  when  $n \rightarrow \infty$ , then from the latter approximate equation there follows the reliability of the system  $\{\omega_i(M)\}$ .

The above statement can be directly carried over to any case of the potential system [17]. Thus, we can show that if  $S$  and  $S_1$  are not tangent to one another, then any potential system is not reliable.

To check the rate at which the Gram determinant approaches zero, we carried out the following numerical experiments for the two-dimensional case.  $S$  is a circle with radius 1 on which we are required to orthonormalize the system of functions  $\{\ln r(x_i, M)\}$  ( $i = 1, 2, \dots, 28$ ), where  $x \in S_1^{(1)}$  is a concentric circle with radius 2. The points  $x_i$  are distributed on  $s_1^{(1)}$  uniformly in  $\frac{\pi}{14}$  intervals. We computed the elements of the Gram determinant of this system. Integration was carried out with an error that did not exceed  $10^{-6}$ . The rank of the determinant with an accuracy up to  $10^{-9}$  was found to be equal to 9, i.e., after sorting out all possible determinants of tenth and higher orders we were unable to detect any among them that were nonmachine-zero (the computations were carried out on a high-speed electronic computer BESM-2). Below we cite the maximal values of  $\overline{G_i}$  of the determinant of  $i^{\text{th}}$  order for  $i = 1, 2, \dots, 9$ :

$$\begin{array}{ccccc}\overline{G}_1=3,8 & \overline{G}_2=9,7 & \overline{G}_3=9,5 & \overline{G}_4=5,9 & \overline{G}_5=0,4 \\ \overline{G}_6=2,7 \cdot 10^{-2} & \overline{G}_7=9,2 \cdot 10^{-4} & \overline{G}_8=2,8 \cdot 10^{-5} & \overline{G}_9=3,7 \cdot 10^{-8} & \end{array}$$

Then the points  $x_i$  were taken uniformly on the concentric circle  $S_1^{(2)}$  with radius 1.1 and interval  $\frac{\pi}{12}$ . The number of points  $n$  was equal to 24. Below we give the respective values for the Gram determinant  $G_i$  ( $i = 1, 2, \dots, 24$ ) for this case /35

$$\begin{array}{cccccc} G_1=3,6 & G_5=80 & G_9=95 & G_{13}=1,8 & G_{17}=1,6 \cdot 10^{-2} & G_{21}=1,5 \cdot 10^{-5} \\ G_2=8,4 & G_6=130 & G_{10}=38 & G_{14}=0,7 & G_{18}=3 \cdot 10^{-3} & G_{22}=2,9 \cdot 10^{-6} \\ G_3=28 & G_7=184 & G_{11}=13 & G_{15}=0,2 & G_{19}=5,1 \cdot 10^{-4} & G_{23}=4,9 \cdot 10^{-7} \\ G_4=48 & G_8=236 & G_{12}=4,7 & G_{16}=8,9 \cdot 10^{-2} & G_{20}=8,6 \cdot 10^{-5} & G_{24}=4,9 \cdot 10^{-8} \end{array}$$

From comparison of the respective values of the Gram determinant it is obvious that in the second case the values of the determinant are much larger which, as will be shown below, makes it possible to orthonormalize the respective system more exactly.

Finally the points  $x_i$  were taken uniformly distributed on a concentric circle  $s_1^{(3)}$  with a radius of 1.05 and the same interval  $\frac{\pi}{12}$ . The number of points is equal to 24. Let us derive the respective values of the Gram determinant  $G_i$  ( $i = 1, 2, \dots, 24$ ):

$$\begin{array}{cccc} G_1=4,1 & G_7=4,7 & G_{13}=2,9 & G_{19}=0,5 \\ G_2=4,7 & G_8=4,6 & G_{14}=2,5 & G_{20}=0,3 \\ G_3=4,8 & G_9=4,3 & G_{15}=2,0 & G_{21}=0,2 \\ G_4=4,9 & G_{10}=4,0 & G_{16}=1,6 & G_{22}=0,1 \\ G_5=4,9 & G_{11}=3,7 & G_{17}=1,2 & G_{23}=0,08 \\ G_6=4,8 & G_{12}=3,3 & G_{18}=0,8 & G_{24}=0,03 \end{array}$$

The results of these numerical experiments show that with the approach of the auxiliary boundary  $S_1$  to the fundamental boundary  $S$ , the corresponding Gram determinant is increased. Let us note that this proof of reliability of the systems of potential functions is substantially based on the constancy of the auxiliary boundary  $S_1$ . Otherwise, from Equation (5.13) we can never derive (5.14), since for some points  $M \in S$  of the functions,  $r(x_{si}, M)$  and  $\xi(M)$  may have one and the same order of smallness.

A second comment touches on the choice of algorithm for orthonormalization of the system  $\{\omega_i(M)\}$ . As was shown in §4, the orthonormalization may be accomplished both according to Formula (4.6) and according to Formula (4.10). It is clear that if the computations according to both formulas are carried out exactly (with a finite number of digits), then the results are also found to be identical. The results will be sufficiently similar also in the case when the system to be orthonormalized is reliable. But, as was shown above, the system  $\{\varphi_i(M)\}$ , just as any potential system, is not reliable. Therefore, for practical application of the approximation method for solving the boundary problems it is extremely important to choose, of the two procedures for orthonormalization, that which will give the more consistent procedure for the computation (will guarantee a larger number of reliable digits). Although the computations according to Formula (4.10) are easier to program on the computer, since for computation of the determinants there already exist prepared standard subprograms, nevertheless, as will be shown below the orthonormalization should be carried out according to Formula (4.6), because the respective algorithm is significantly more consistent relative to rounding off errors. /36

We shall analyze the normalized functions  $\omega_i(M)$  as vectors (with origin at 0) in the space  $L_2$  and denote by  $\alpha_k$  the angle between the vector  $\omega_k(M)$  and the hyperplane passing through the vectors  $\omega_1, \omega_2, \dots, \omega_{k-1}$ . It is known [16], that the determinant (4.11) is equal to the square of the volume of the parallelepiped constructed on the vectors  $\omega_1, \omega_2, \dots, \omega_n$ .

$$G_n = \prod_{i=1}^n \sin^2 \alpha_i.$$

Thus, for the denominator of Formula (4.10), we find

$$d_n = \sqrt{G_{n-1} \cdot G_n} = \sin \alpha_n \prod_{i=1}^{n-1} \sin^2 \alpha_i.$$

As far as the following sum is concerned

$$\sum_{k=1}^{n-1} (\omega_n, \varphi_k) \varphi_k, \quad (5.15)$$

it represents [16] the projection of the element  $\omega_n$  on the subspace of the vectors  $\phi_1, \phi_2, \dots, \phi_{n-1}$  or on the strength of the equivalence of the subspaces of the vectors  $\phi_1, \phi_2, \dots, \phi_n$  and  $\omega_1, \omega_2, \dots, \omega_n$ , (5.15) represents the projection  $\omega_n$  on the subspace of the vectors  $\omega_1, \omega_2, \dots, \omega_n$ . Hence, it is obvious that the denominator in Formula (4.6) is equal to

$$\sqrt{\int \int_s \bar{\varphi}_n^2 ds} = \sin \alpha_n.$$

For orthonormalization of  $n$  elements we find it necessary to divide by

$$\bar{d}_n = \prod_{i=1}^n \sin \alpha_i.$$

For the ratio  $d_n : \bar{d}_n$  we find

$$d_n : \bar{d}_n = \prod_{i=1}^{n-1} \sin \alpha_i = \bar{d}_{n-1}. \quad (5.16)$$

From the latter expression it is clear that Formula (4.6) gives a significantly more consistent computational procedure than does Formula (4.10).

We attempted to orthonormalize the system  $\{\ln r(x_i, M)\}$ ,  $x \in S_1^{(1)}$  ( $i = 1, 2, \dots, 28$ ) with the aid of Formula (4.6); however, an emergency halt took place in the machine. It was found that this occurred when dividing by  $\int \bar{\varphi}_{21}^2 ds$ . This fact (taking into account that all the Gram determinants of tenth order are equal to zero) agrees well with Formula (5.16) from which it follows that Formula (4.6) can be used for the fixed number of digits with which the computations are carried out, to orthonormalize approximately twice the number of functions as with Formula (4.10). It is clear that if the orthonormalization of a sufficiently reliable system can be carried out both according to Formula (4.6) and to Formula (4.10), then this latter gives a significantly rougher result. To confirm this we carried out orthonormalization of the system  $\{\ln r(x_i, M)\}$ ,  $x \in S_1^{(2)}$  ( $i = 1, 2, \dots, 24$ ). Table 1 gives the coefficients of orthonormalization  $A_{22, i}^{(1)}$  for the function  $\phi_{22}$  obtained from Formula (4.10), and the same coefficients  $A_{22, i}^{(2)}$  obtained from (4.6). /37

In the third and fourth columns are given  $\alpha_{22,i}^{(1)} = \int \varphi_{22}^{(1)} \varphi_i^{(1)} ds$  and  $\alpha_{22,i}^{(2)} = \int \varphi_{22}^{(2)} \varphi_i^{(2)} ds$ , where  $\varphi_i^{(1)}$  and  $\varphi_i^{(2)}$  are orthonormalized functions obtained respectively with the aid of Formulas (4.10) and (4.6).

TABLE 1

$i$	$A_{22,i}^{(1)}$	$A_{22,i}^{(2)}$	$\alpha_{22,i}^{(1)}$	$\alpha_{22,i}^{(2)}$
1	1.5629	1.5747	1.37	0.9993
2	-1.7303	-1.7267	0.15	0.0080
3	0.7977	0.7967	-0.02	-0.0030
4	-0.1783	-0.1772	-0.13	0.0009
5	0.1160	0.1195	-0.20	-0.0007
6	0.0325	0.0258	-0.22	-0.0007
7	0.0264	0.0541	-0.25	-0.0005
8	0.1276	0.0451	-0.20	-0.0005
9	-0.2289	0.0477	-0.36	-0.0005
10	0.7624	0.0468	0.13	-0.0004
11	-0.9413	0.0471	-0.37	-0.0004
12	0.6365	0.0469	-0.17	-0.0003
13	0.1307	0.0470	0.06	-0.0003
14	-0.3814	0.0469	-0.19	-0.0003
15	0.3762	0.0472	-0.07	-0.0003
16	-0.0596	0.0465	-0.05	-0.0003
17	0.0032	0.0492	-0.10	-0.0003
18	0.3059	0.0412	0.05	-0.0006
19	-0.6559	0.0685	-0.31	0.0040
20	1.0015	-0.0156	0.17	0.0020
21	-0.4316	0.2664	0.00	0.0020
22	-0.1754	-0.3722	0.00	-0.0009

We see the Formula (4.6) gives a significantly more exact orthonormalization than does Formula (4.10).

In Reference [16], the phenomenon of "stability" was first investigated for systems of functions, and the class of systems was indicated that are termed reliable for which the Ritz method remains stable. Later [19] S. G. Mikhlin indicated a significantly wider class (strongly minimal systems), for which the Ritz method retains its stability. The system of functions  $\{\omega_i\}$  is strongly minimal in  $H_A$  [20] if the least eigenvalue of the  $n^{\text{th}}$  order Ritz matrix is bounded below by a positive constant which is independent of  $n$ . Since the potential systems which we investigated may be

used as a coordinate system in the Ritz method, then it is interesting to know /38 if they are strongly minimal. Let  $S$  and  $S_1$  be concentric circles and the points  $x_i (i = 1, 2, \dots, 2N)$ , where  $N$  is even, be distributed uniformly on  $S_1$ . The following assumption is valid. The systems  $\{\ln r(x_i, M)\}$  and  $\left\{\frac{\partial}{\partial n} \ln r(x_i, M)\right\}$  are not strongly minimal in  $H_E$  if  $E$  is an identity operator. Since when  $H_E$  ( $H_E$  coincides with  $L_2$ ), the Ritz matrix coincides with the Gram determinant, then it is confirmed that for any  $\epsilon > 0$  we find a value of  $N$  such that the eigenvalue (with the smallest modulus) of the Gram determinant of  $2N^{\text{th}}$  order is less than  $\epsilon$ . In view of the identify of the proofs, we can give a proof for the system  $\{\ln r(x_i, M)\}$ . We must prove that the least eigenvalue of matrix (4.11) is as near to zero as desired.

It is easy to prove that for the given case [ $S$  and  $S_1$  are concentric circles,  $x_i (i = 1, 2, \dots, 2N)$  is distributed unifromly on  $S_1$ ] the following equations are valid:

$$\begin{aligned} \int \omega_i \omega_{<i+k>} ds &= \int \omega_i \omega_{i-k} ds, \\ \int \omega_j \omega_r ds &= \int \omega_\xi \omega_\eta ds \quad \text{for } |j-r| = |\xi-\eta|, \end{aligned}$$

$$k=0, 1, \dots, 23; i, j, r, \xi, \eta=1, 2, \dots, 24; k < i; <i+k> = i+k \bmod (n-1)$$

and therefore the Matrix (4.11) has the following form:

$$\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \dots & a_1 \end{vmatrix}$$

where  $a_s = a_{n+2-s} - x$  ( $s = 2, 3, \dots, N$ ),

$$a_s = \int \ln r(x_j, M) \ln r(x_{j+s-1}, M) ds_{\mu} \quad (5.17)$$

The eigenvalues of this matrix are equal [18] to

$$\lambda_k = \sum_{i=1}^n a_i e_k^{i-1} \quad (k=0, 1, \dots, n-1), \quad (5.18)$$

where  $\varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ . When  $k = N$  we find

$$\lambda_N = \sum_{i=1}^n (-1)^{i-1} a_i. \quad (5.19)$$

Taking (5.17) into account, we find

$$\begin{aligned} a_{i+1} - a_i &= \int \ln r(x_j, M) [\ln r(x_{j+i}, M) - \ln r(x_{j+i-1}, M)] ds = \\ &= \frac{2\pi R}{n} \int \ln r(x_j, M) \left[ \frac{\partial}{\partial S_1} \ln r(x_{j+i}, M) + O\left(\frac{1}{n}\right) \right] ds, \end{aligned} \quad (5.20)$$

where  $R$  is the radius of the circle  $S_1$ ,  $\frac{\partial}{\partial S_1}$  is the derivative along the /39  
tangent to the circle  $S_1$ . The left-hand side of (5.20) does not depend on  $j$ ;  
therefore, we will assume that  $j = 1$

$$a_{i+1} - a_i = \frac{2\pi R}{n} \int \ln r(x_1, M) \frac{\partial}{\partial S_1} \ln r(x_{i+1}, M) + O\left(\frac{1}{n^2}\right)$$

or

$$a_2 - a_1 = \frac{2\pi R}{n} \int \ln r(x_1, M) \frac{\partial}{\partial S_1} \ln r(x_2, M) ds + O\left(\frac{1}{n^2}\right)$$

$$a_4 - a_3 = \frac{2\pi R}{n} \int \ln r(x_1, M) \frac{\partial}{\partial S_1} \ln r(x_4, M) ds + O\left(\frac{1}{n^2}\right)$$

.....

$$a_{2N-2} - a_{2N-3} = \frac{2\pi R}{n} \int \ln r(x_1, M) \frac{\partial}{\partial S_1} \ln r(x_{2N-2}, M) ds + O\left(\frac{1}{n^2}\right)$$

$$a_{2N} - a_{2N-1} = \frac{2\pi R}{n} \int \ln r(x_1, M) \frac{\partial}{\partial S_1} \ln r(x_{2N}, M) ds + O\left(\frac{1}{n^2}\right).$$

Substituting these latter equations into (5.19), taking the evenness of  $N$  into account and the trivial equations

$$\begin{aligned} \int \ln r(x_1, M) \frac{\partial}{\partial S_1} \ln r(x_{2s}, M) ds &= - \int \ln r(x_1, M) \frac{\partial}{\partial S_1} \ln r(x_{n+2-2s}, M) ds \\ &\quad \left( s = 1, 2, \dots, \frac{N}{2} \right); \end{aligned}$$

we find the asymptotic equations

$$\lambda_N = O\left(\frac{1}{n}\right). \quad (5.21)$$

We can show that for even  $N/2$  the absolute value of the characteristic number  $\lambda_{\frac{N}{2}}$  will be as small as desired. In fact

$$\lambda_{\frac{N}{2}} = \sum_{k=1}^n (i)^k a_k = \sum_{k=1}^{\frac{N}{2}} [(i a_{4k-3} - a_{4k-1}) - (a_{4k} - a_{4k-2})].$$

The real part of the complex number  $\lambda_{\frac{N}{2}}$  is equal to zero because of the accuracy of  $N/2$  and of the trivial equations

$$a_{4k-2} = a_{4(\frac{N}{2}+1-k)}; \quad a_{4k} = a_{4(\frac{N}{2}+1-k)-2} \quad \left(k = 1, 2, \dots, \frac{N}{2}\right),$$

and the imaginary part

$$I_m \left( \lambda_{\frac{N}{2}} \right) = \sum_{k=1}^N (-1)^{k+1} a_{2k-1}.$$

Proof of the asymptotic equations (6)

$$I_m \left( \lambda_{\frac{N}{2}} \right) = O \left( \frac{1}{n} \right)$$

/40

is completely analogous to the proof of (5.21).

The potential systems will be called discontinuous potential systems, if  $S_1$  is consistent with  $S$ , i.e., if the points  $x_i$  are distributed on the fundamental boundary  $S$ . The following assumptions are valid. The system of functions  $\{\ln r(x_i, M)\}$ , where the points  $x_i$  are distributed everywhere densely on  $S$ , are linearly independent and complete in the space  $L^2(s)$ .

The system of functions  $\{\omega_i(M)\}$ , where  $\omega_0$  is a nonzero constant,

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(6) From the constructions of the potential systems it is clear that they are not minimal and consequently not strongly minimal. The proved assumption is interesting in that it gives to the asymptotics the least eigenvalue of the Gram matrix.

$\omega_i(M) = \frac{\partial}{\partial n_x} \ln r(x_i, M)$  ( $i=1, 2, \dots$ ),  $x_i$  distributed everywhere densely on  $S$ , are linearly independent and complete in  $L^2(s)$ .

The linear independence of these systems is obvious and is proved analogously to the linear independence of the potential systems. For proof of the completeness it is sufficient to prove that the operators

$$\int_s \varphi(M) \ln r(x, M) ds_M, \quad \int_s \varphi(M) \frac{\partial}{\partial n} \ln r(x, M) ds_M \quad (5.22)$$

transform ( $\varphi \in L^2(s)$ ) to space  $L^2(s)$  in  $C$ .

Further arguments are completely analogous to the discussions in proving completeness of the potential systems (see the proof of Theorem 4.4).

Let us analyze the integral operation

$$y(x) = \int_s \varphi(M) k(x, M) ds_M, \quad (5.23)$$

and assume that all singularities of the kernel  $k(x, M)$  are concentrated on the diagonal, i.e., when  $x = M$ . We know [12] that if the kernel of Operation (5.23) satisfies the conditions

$$\left\{ \int_s | |\text{grad}_x k(x, M)| | [r(x, M)]^{1-\mu} ]^p ds_M \right\}^{\frac{1}{p}} \leq E, \quad (5.24)$$

$$\left\{ \int_s \left[ \frac{|k(x, M)|}{[r(x, M)]^\mu} \right]^p ds_M \right\}^{\frac{1}{p}} \leq F, \quad (5.25)$$

where by  $\text{grad}_x$  we denote the gradient computed according to the variable  $x$ , then the integral Operation (5.23) maps the space  $L^{p'}$  (and any  $L^q$  when  $q \geq p'$ ) /41 into the Lipschitz space  $\text{Lip } \mu$  with the index  $\mu$ , where  $p'$  is the adjoint index ( $1/p + 1/p' = 1$ ).

It is easy to see that for the kernels of the integral operations (5.22), Conditions (5.24) - (5.25) are satisfied in the case of piecewise-smooth contours when  $\rho = \rho' = 2$  and  $\mu < \frac{1}{2}$ .

The system of functions  $\left\{ \frac{\partial}{\partial n} \ln r(x_i, M) \right\}$  is used [1-3] for solving the Neumann problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } G, \\ \frac{\partial u}{\partial n} \Big|_S &= \psi(y). \end{aligned} \quad (5.26)$$

The approximate solution to Problem (5.26) can be obtained from the expansion of the function

$$\varphi(y) = \int_{y_0}^y \psi(y) ds_y$$

for the system  $\{\ln r(x_i, y)\}$

$$\varphi(y) \approx \sum_{k=1}^N a_k \ln r(x_k, y).$$

In fact, the approximate solution to Problem (5.26) has the form

$$u \approx \sum_{k=1}^N a_k \operatorname{Arg}(z - z_k) = \sum_{k=1}^N a_k \operatorname{arctg} \frac{y^{(2)} - x_k^{(2)}}{y^{(1)} - x_k^{(1)}},$$

where  $x_k^{(1)}$ ,  $x_k^{(2)}$  and  $y^{(1)}$ ,  $y^{(2)}$  are the coordinates of the points  $x_k$  and  $y$ , respectively,

$$z = y^{(1)} + i y^{(2)}, \quad z_k = x_k^{(1)} + i x_k^{(2)}.$$

Let us analyze the system of three-dimensional discontinuous functions

$$\left\{ \frac{1}{r(x, M)} \right\}, \quad (5.27)$$

where  $x_i$  are distributed on the surface  $S$ . The following assumption is valid. The system of Functions (5.27) is complete in  $L^{p'}$  for  $p' = \frac{2-\alpha}{1-\alpha}$  for any

$\alpha > 0$ , and consequently, on the strength of Theorem 3.8, is closed in  $L^p$  ( $p=2-\alpha$ ). For proof let us mention that Conditions (5.24), (5.25) are satisfied for  $k(x, M) = \frac{1}{r(x, M)}$  when  $p=2-\alpha$ ,  $\mu < \frac{\alpha}{2-\alpha}$  and any  $\alpha > 0$ . Therefore, the integral operation

$$\int \frac{1}{r(x, M)} \varphi(M) ds_M \quad (5.28)$$

transforms the space  $L^{p'}$   $p' = \frac{2-\alpha}{1-\alpha}$  for any  $\alpha > 0$  to a Lipschitz space  $Lip$  /42  
 $\mu$ ,  $\mu < \frac{\alpha}{2-\alpha}$ .

Let the function  $\varphi(M) \in L^{p'}$  be orthogonal to all functions of System (5.27). Let us prove that  $\varphi(M)=0$ . We analyze the continuous Function (5.28). On the set of points  $x_i$  — which is everywhere dense on  $s$ , this function takes zero values, and therefore, it is equal to absolute zero on  $s$  and consequently (on the strength of the harmonicity) in all three-dimensional space. But, from Reference [12, it follows then that the density  $\varphi(M)=0$ . Taking into account Theorem 3.8, we find that System (5.27) is closed in  $L_p$  when  $L$   $p=2-\alpha$ .

#### §6. Solution to Boundary Value Problems with the Aid of Nonorthogonal Series

References [25, 26] give one method for determining the expansion coefficients for a system of nonorthogonal functions. The idea of this method involves the following (it is described in detail in the next section). Let us seek the expansion  $\sum e_k g_k(s)$  of the functions  $F(s)$  according to the normalized system  $\{g_k(s)\}$  ( $k=1, 2, \dots$ ). Let us introduce the symbols

$$F_n(s) = - \sum_{i=0}^n e_i g_i(s), \quad F_0(s) = g_0(s) = F(s), \quad e_0 = -1. \quad (6.1)$$

To compute  $e_k$  in Reference [25, 26] it is proposed to use the formulas

$$e_k = \int_s F_{k-1} g_k ds. \quad (6.2)$$

This method of computing the expansion coefficients directly may be used in the method of generalized Fourier series, since in this latter case the function to be expanded is known. The numerical experiments showed, however, a significantly slower convergence of the method of generalized series when Formula (6.2) is used in comparison with the method of generalized orthogonal Fourier series.

In the present section we shall show [33] such a modification of the method of functional equations on the basis of Formulas (6.1), (6.2), the application of which will not encounter the difficulties mentioned in References [22, 24].

For concreteness let us analyze the method of functional equations for the two-dimensional internal Dirichlet boundary value problem.

$$\begin{aligned} \Delta u &= 0 \text{ in } G, \\ u|_s &= f(s), \end{aligned} \tag{6.3}$$

where  $s$  is the boundary of the region  $G$ ;  $f(s)$  is a given function. The essence of the method of V. D. Kupradze involves the following. In the sense of the metric  $L_2(s)$  the best expansion is constructed for the normal derivative  $\frac{\partial u}{\partial n}|_s \varphi(s)$  of the unknown function according to the functions of the complete and linearly independent system  $\{\ln r(x_k, s)\} = \{\omega_k(s)\}$ , where  $r(x_k, s)$  is the distance between points  $x_k$  and  $s$ ;  $\{x_k\}$  is a set of points distributed everywhere densely on the auxiliary boundary  $s$ , which completely includes the region  $G$ . If we have such an expansion, the solution to Problem (6.3) can be found from the fundamental integral equation of the theory of harmonic functions

$$u(x) = \int_s \varphi(s) \ln r(x, s) ds + \int_s f(s) \frac{\partial}{\partial n} \ln r(x, s) ds, \quad x \in G. \tag{6.4}$$

We know [10] that for minimality of the expression  $\|\varphi(s) - \sum_{k=1}^n a_k \omega_k(s)\|_{L_2}$  with respect to the coefficients  $a_k$  it is necessary and sufficient that  $a_k$  be solutions to the system

$$\sum_{k=1}^n a_k \int_s \omega_k(s) \omega_i(s) ds = \int_s \varphi(s) \omega_i(s) ds \quad (i=1, 2, \dots, n), \quad (6.5)$$

but on the strength of the Green identity

$$\int_s \varphi(s) \omega_i(s) ds = \int_s f(s) \frac{\partial}{\partial n} \omega_i(s) ds \quad (6.6)$$

and System (6.5) takes the form

$$\sum_{k=1}^n a_k \int_s \omega_k(s) \omega_i(s) ds = \int_s f(s) \frac{\partial}{\partial n} \omega_i(s) ds. \quad (6.7)$$

If we first obtain the orthonormalized system  $\{\varphi_k(s)\} = \left\{ \sum_{i=1}^k A_{k,i} \omega_i(s) \right\}$ , where  $A_{k,i}$  are the coefficients of orthonormalization, then in the sense of  $L_2$  the best expansion of the function  $\varphi(s)$  will be, as we know, expansion in the Fourier series

$$\sum_{k=1}^n b_k \varphi_k(s) = \sum_{k=1}^n b_k \sum_{i=1}^k A_{k,i} \omega_i(s) = \sum_{k=1}^n c_k \omega_k(s),$$

where  $b_k$  are the Fourier coefficients of the function  $\varphi(s)$ , and

$$c_k = \sum_{i=k}^n A_{i,k} b_i. \quad (6.8)$$

On the strength of the strict normalization [10] of the space  $L_2$  and consequently the uniqueness of the generalized polynomial of the best approximation, the solutions  $a_k$  of the System (6.7) and  $c_k$  from (6.8) must be identical to  $a_k = c_k$  ( $k = 1, 2, \dots, n$ ).

Thus, the method of functional equations may be analyzed as a combination of the variational method (for the normal derivative of the unknown function) /44 by using the Green Formula (6.4).

It is clear that if we find the expansion  $\sum_{k=1}^n d_k \omega_k(s)$  not of the normal derivative of the unknown function, but of its boundary value  $f(s)$ , then the

approximate solution at any point  $M$  of the region  $G$  may be found directly from the expression (method of generalized Fourier series)

$$u(M) = \sum_{k=1}^n d_k \ln r(x_k, M),$$

and the necessity of carrying out the quadratures associated with the use of Formula (6.4) is eliminated.

However, numerous numerical experiments<sup>(7)</sup> have shown that for one and the same number of functions which participate in the expansion, the method of V. D. Kupradze gives significantly more accurate results than the method of generalized Fourier series. Therefore, the supplemental computation associated with using the Green formula is completely justified. For illustration, below we cite the errors in solving the Dirichlet problem for the function  $u = \arctan y-2/x-2$  in the case of an ellipse with semiaxes  $a = 1$ ,  $b = 0.75$ . Table 2 gives these errors for  $x_k \in s_1^{(2)}$  and  $n = 24$  at the mesh points with an interval of  $h = 0.1$ . The upper left number in each point corresponds to the error in V. D. Kuproadze's method. Following this is the error in the method of generalized Fourier series, and below is the error in the method of finite differences obtained with the standard program [31, 32] (the simplest approximation of the Laplace operator and the Kollats deflection) with an interval of  $h = 0.1$ .

However, with an increase in the number of orthonormalizable functions or when the auxiliary boundary  $S_1$  moves away from the fundamental boundary  $s$ , the corresponding Gram determinant for the unreliable system  $\{\omega_k(s)\}$  approaches zero, which makes it practically impossible to carry out orthonormalization. As noted in §5 with concentric circles  $S$  and  $S_1$  having radii  $r = 1$  and  $r_1 = 2$ , use of the matrix method of orthonormalization will not allow orthonormalizing

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<sup>(7)</sup> The numerical experiments were carried out with the aid of standard programs [27,28] compiled in the Department of Numerical Methods of the Computer Center, Academy of Sciences, Georgian SSR.



TABLE 2

0.7	-17746	-27755	-36391	-44700	-52688	-60359	-67714	-74757	-81488	-87817	-93746	-99275	-104404	-109133	-113462	-117391	-120920	-124049	-126778	-129107	-131036	-132565	-133794	-134623	-135152	-135381	-135310	-134939	-134268	-133297	-132026	-130455	-128584	-126413	-123942	-121171	-118100	-114829	-111358	-107687	-103816	-99745	-95474	-90903	-86132	-81161	-76090	-70919	-65648	-60277	-54806	-49235	-43564	-37793	-31922	-25951	-19880	-13709	-7538	-1317	4804	10975	17946	24817	31588	38259	44830	51291	57642	63893	69944	75795	81446	86897	92148	97199	102050	106701	111162	115433	119504	123375	127046	130517	133788	136859	139730	142391	144852	147113	149174	151035	152696	154157	155418	156479	157340	157991	158452	158723	158804	158695	158396	157907	157228	156359	155290	154031	152582	150943	149114	147095	144876	142457	139838	137019	133990	130751	127302	123653	119804	115755	111506	107057	102408	97539	92370	86901	81132	75063	68694	62025	55156	48087	40818	33349	25680	17911	9942	1673	-1116	-2885	-4646	-6397	-8138	-9869	-11590	-13301	-15002	-16693	-18374	-20045	-21706	-23357	-25008	-26649	-28280	-29901	-31512	-33113	-34704	-36285	-37856	-39417	-40968	-42509	-44040	-45561	-47072	-48573	-50064	-51545	-53016	-54477	-55928	-57369	-58800	-60221	-61632	-63033	-64424	-65805	-67176	-68537	-69888	-71219	-72540	-73851	-75152	-76443	-77724	-78995	-80256	-81507	-82748	-83969	-85180	-86381	-87572	-88753	-89924	-91085	-92236	-93377	-94508	-95629	-96740	-97841	-98932	-100013	-101084	-102145	-103196	-104237	-105268	-106289	-107290	-108281	-109262	-110233	-111194	-112145	-113086	-114017	-114938	-115849	-116750	-117641	-118522	-119393	-120254	-121105	-121946	-122777	-123598	-124409	-125210	-125991	-126752	-127503	-128244	-128975	-129696	-130407	-131108	-131799	-132470	-133131	-133782	-134433	-135074	-135705	-136326	-136937	-137538	-138129	-138710	-139281	-139842	-140393	-140934	-141465	-141986	-142497	-142998	-143489	-143970	-144441	-144892	-145333	-145764	-146185	-146596	-147007	-147408	-147799	-148180	-148551	-148912	-149263	-149604	-149935	-150256	-150567	-150868	-151159	-151440	-151711	-151972	-152223	-152464	-152695	-152916	-153127	-153328	-153519	-153690	-153851	-154002	-154143	-154274	-154395	-154506	-154607	-154698	-154779	-154850	-154911	-154962	-155013	-155064	-155115	-155166	-155217	-155268	-155319	-155370	-155421	-155472	-155523	-155574	-155625	-155676	-155727	-155778	-155829	-155880	-155931	-155982	-156033	-156084	-156135	-156186	-156237	-156288	-156339	-156390	-156441	-156492	-156543	-156594	-156645	-156696	-156747	-156798	-156849	-156890	-156941	-156992	-157043	-157094	-157145	-157196	-157247	-157298	-157349	-157390	-157441	-157492	-157543	-157594	-157645	-157696	-157747	-157798	-157849	-157890	-157941	-157992	-158043	-158094	-158145	-158196	-158247	-158298	-158349	-158390	-158441	-158492	-158543	-158594	-158645	-158696	-158747	-158798	-158849	-158890	-158941	-158992	-159043	-159094	-159145	-159196	-159247	-159298	-159349	-159390	-159441	-159492	-159543	-159594	-159645	-159696	-159747	-159798	-159849	-159890	-159941	-159992	-160043	-160094	-160145	-160196	-160247	-160298	-160349	-160390	-160441	-160492	-160543	-160594	-160645	-160696	-160747	-160798	-160849	-160890	-160941	-160992	-161043	-161094	-161145	-161196	-161247	-161298	-161349	-161390	-161441	-161492	-161543	-161594	-161645	-161696	-161747	-161798	-161849	-161890	-161941	-161992	-162043	-162094	-162145	-162196	-162247	-162298	-162349	-162390	-162441	-162492	-162543	-162594	-162645	-162696	-162747	-162798	-162849	-162890	-162941	-162992	-163043	-163094	-163145	-163196	-163247	-163298	-163349	-163390	-163441	-163492	-163543	-163594	-163645	-163696	-163747	-163798	-163849	-163890	-163941	-163992	-164043	-164094	-164145	-164196	-164247	-164298	-164349	-164390	-164441	-164492	-164543	-164594	-164645	-164696	-164747	-164798	-164849	-164890	-164941	-164992	-165043	-165094	-165145	-165196	-165247	-165298	-165349	-165390	-165441	-165492	-165543	-165594	-165645	-165696	-165747	-165798	-165849	-165890	-165941	-165992	-166043	-166094	-166145	-166196	-166247	-166298	-166349	-166390	-166441	-166492	-166543	-166594	-166645	-166696	-166747	-166798	-166849	-166890	-166941	-166992	-167043	-167094	-167145	-167196	-167247	-167298	-167349	-167390	-167441	-167492	-167543	-167594	-167645	-167696	-167747	-167798	-167849	-167890	-167941	-167992	-168043	-168094	-168145	-168196	-168247	-168298	-168349	-168390	-168441	-168492	-168543	-168594	-168645	-168696	-168747	-168798	-168849	-168890	-168941	-168992	-169043	-169094	-169145	-169196	-169247	-169298	-169349	-169390	-169441	-169492	-169543	-169594	-169645	-169696	-169747	-169798	-169849	-169890	-169941	-169992	-170043	-170094	-170145	-170196	-170247	-170298	-170349	-170390	-170441	-170492	-170543	-170594	-170645	-170696	-170747	-170798	-170849	-170890	-170941	-170992	-171043	-171094	-171145	-171196	-171247	-171298	-171349	-171390	-171441	-171492	-171543	-171594	-171645	-171696	-171747	-171798	-171849	-171890	-171941	-171992	-172043	-172094	-172145	-172196	-172247	-172298	-172349	-172390	-172441	-172492	-172543	-172594	-172645	-172696	-172747	-172798	-172849	-172890	-172941	-172992	-173043	-173094	-173145	-173196	-173247	-173298	-173349	-173390	-173441	-173492	-173543	-173594	-173645	-173696	-173747	-173798	-173849	-173890	-173941	-173992	-174043	-174094	-174145	-174196	-174247	-174298	-174349	-174390	-174441	-174492	-174543	-174594	-174645	-174696	-174747	-174798	-174849	-174890	-174941	-174992	-175043	-175094	-175145	-175196	-175247	-175298	-175349	-175390	-175441	-175492	-175543	-175594	-175645	-175696	-175747	-175798	-175849	-175890	-175941	-175992	-176043	-176094	-176145	-176196	-176247	-176298	-176349	-176390	-176441	-176492	-176543	-176594	-176645	-176696	-176747	-176798	-176849	-176890	-176941	-176992	-177043	-177094	-177145	-177196	-177247	-177298	-177349	-177390	-177441	-177492	-177543	-177594	-177645	-177696	-177747	-177798	-177849	-177890	-177941	-177992	-178043	-178094	-178145	-178196	-178247	-178298	-178349	-178390	-178441	-178492	-178543	-178594	-178645	-178696	-178747	-178798	-178849	-178890	-178941	-178992	-179043	-179094	-179145	-179196	-179247	-179298	-179349	-179390	-179441	-179492	-179543	-179594	-179645	-179696	-179747	-179798	-179849	-179890	-179941	-179992	-180043	-180094	-180145	-180196	-180247	-180298	-180349	-180390	-180441	-180492	-180543	-180594	-180645	-180696	-180747	-180798	-180849	-180890	-180941	-180992	-181043	-181094	-181145	-181196	-181247	-181298	-181349	-181390	-181441	-181492	-181543	-181594	-181645	-181696	-181747	-181798	-181849	-181890	-181941	-181992	-182043	-182094	-182145	-182196	-182247	-182298	-182349	-182390	-182441	-182492	-182543	-182594	-182645	-182696	-182747	-182798	-182849	-182890	-182941	-182992	-183043	-183094	-183145	-183196	-183247	-183298	-183349	-183390	-183441	-183492	-183543	-183594	-183645	-183696	-183747	-183798	-183849	-183890	-183941	-183992	-184043	-184094	-184145	-184196	-184247	-184298	-184349	-184390	-184441	-184492	-184543	-184594	-184645	-184696	-184747	-184798	-184849	-184890	-184941	-184992	-185043	-185094	-185145	-185196	-185247	-185298	-185349	-185390	-185441	-185492	-185543	-185594	-185645	-185696	-185747	-185798	-185849	-185890	-185941	-185992	-186043	-186094	-186145	-186196	-186247	-186298	-186349	-186390	-186441	-186492	-186543	-186594	-186645	-186696	-186747	-186798	-186849	-186890	-186941	-186992	-187043	-187094	-187145	-187196	-187247	-187298	-187349	-187390	-187441	-187492	-187543	-187594	-187645	-187696	-187747	-187798	-187849	-187890	-187941	-187992	-188043	-188094	-188145	-188196	-188247	-188298	-188349	-188390	-188441	-188492	-188543	-188594	-188645	-188696	-188747	-188798	-188849	-188890	-188941	-188992	-189043	-189094	-189145	-189196	-189247	-189298	-189349	-189390	-189441	-189492	-189543	-189594	-189645	-189696	-189747	-189798	-189849	-189890	-189941	-189992	-190043	-190094	-190145	-190196	-190247	-190298	-190349	-190390	-190441	-190492	-190543	-190594	-190645	-190696	-190747	-190798	-190849	-190890	-190941	-190992	-191043	-191094	-191145	-191196	-191247	-191298	-191349	-191390	-191441	-191492	-191543	-191594	-191645	-191696	-191747	-191798	-191849	-191890	-191941	-191992	-192043	-192094	-192145	-192196	-192247	-192298	-192349	-192390	-192441	-192492	-192543	-192594	-192645	-192696	-192747	-192798	-192849	-192890	-192941	-192992	-193043	-193094	-193145	-193196	-193247	-193298	-193349	-193390	-193441	-193492	-193543	-193594	-193645	-193696	-193747	-193798	-193849	-193890	-193941	-193992	-194043	-194094	-194145	-194196	-194247	-194298	-194349	-194390	-194441	-194492	-194543	-194594	-194645	-194696	-194747	-194798	-194849	-194890	-194941	-194992	-195043	-195094	-195145	-195196	-195247	-195298	-195349	-195390	-195441	-195492	-195543	-195594	-195645	-195696	-195747	-195798	-195849	-195890	-195941	-195992
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146

		-11	-10	-9	-8	-7			
		16	-15	-12	10	-7			
	-9	-9	-8	-7	-6	-4	-3		
	-14	12	-10	-8	-6	3	1		
-8	-7	-6	-6	-4	-3	-2	0	2	
-14	-12	-10	-7	-5	-2	1	3	7	
-6	-6	-4	-3	-2	-1	1	3	4	
-11	-9	-7	-4	-2	1	4	8	11	
-5	-3	-2	-1	0	1	3	5	7	
-9	-6	-4	-1	1	4	7	11	15	
-2	-2	-1	1	2	4	5	7	9	
-6	-4	1	1	4	-7	10	14	17	
-1	0	-1	2	4	5	7	9	11	
-4	-2	1	3	6	9	13	16	20	
	2	3	4	6	7	9	10		
	0	3	6	9	12	15	19		
		4	5	7	9	10			
		5	8	11	14	17			

147

147

147

happen if the necessity arises for harmonically continuing the solution to Problem (6.1) sufficiently far from  $s$ ), then in this case it is feasible to use the modified method of functional equations.

Analogously we can modify the method of functional equations also for other boundary value problems.

Let us indicate one possible reason for the high degree of accuracy in the method of functional equations in comparison with the method of generalized Fourier series.

The error  $\varepsilon(x)$  of the approximate solution to Problem (6.3) at point  $x$  in the method of generalized Fourier series is equal to

$$\varepsilon(x) = \int_s \left( f(s) - \sum_{k=1}^n a_k w_k(s) \right) \frac{\partial G(x, s)}{\partial n} ds,$$

where  $G(x, s)$  is the Green function, and for its computation by using the Buniakowski-Schwarz inequality, we obtain

$$|\varepsilon(x)| < \left\{ \int_s \left[ f(s) - \sum_{k=1}^n d_k w_k(s) \right]^2 ds \right\}^{\frac{1}{2}} \left\{ \int_s \left[ \frac{\partial G(x, s)}{\partial n} \right]^2 ds \right\}^{\frac{1}{2}}. \quad (6.9)$$

For computation of the error  $\varepsilon_1(x)$  in the method of functional equations

$$|\varepsilon_1(x)| < \left\{ \int_s \left[ \varphi(s) - \sum_{k=1}^n a_k w_k(s) \right]^2 ds \right\}^{\frac{1}{2}} \left\{ \int_s \left[ \ln r(x, s) \right]^2 ds \right\}^{\frac{1}{2}} \quad (6.10)$$

We know that the Green function  $G(x, s)$  has a logarithmic singularity, and therefore, in the equation

$$\left\| f(s) - \sum_{k=1}^n d_k w_k(s) \right\|_{L_2} = \left\| \varphi(s) - \sum_{k=1}^n a_k w_k(s) \right\|_{L_2}$$

computation of (6.9) allows a larger value than computation of (6.10).

In conclusion let us say a few words about computation of the error in the method of functional equations. We know [34] that fundamental problems in the theory of approximate methods are encountered in the following sequence:

1. Construction of an algorithm.
2. Establishment of a convergence.
3. Computation of the rapidity of convergence.

/48

As shown above, the first two problems for V. D. Kupradze's method are solved.

For the error  $\varepsilon^{(N)}(x) = u(x) - u^{(N)}(x)$  of the approximate solution to the Dirichlet problem, we find the following expression:

$$\varepsilon_u^{(N)}(x) = \frac{1}{4\pi} \iint \frac{1}{r(x, y)} [\varphi(y) - \varphi^{(N)}(y)] ds, \quad (6.11)$$

where  $\varphi^{(N)}$  is the finite generalized Fourier sum for the function  $\varphi(y)$  according to a certain complete orthonormalized system of functions. Computation of the difference

$$||\varphi(y) - \varphi^{(N)}(y)||$$

is a problem of general harmonic analysis and for its solution we must generalize the theorem of D. Jackson [10] (see, for example, the generalized theorem of Stone-Weierstrass [35]). We know [34] that a real a priori computation can be obtained only in a quite limited number of problems, and therefore the a posteriori computations are of no less significance, all the more since such a computation can be obtained by machine methods and it may be [34] used for automatically changing future computational programs.

If we assume that  $\phi(s)$  has a finite derivative with respect to the normal (the corresponding sufficient conditions are given in [36]) from the boundary conditions and the Taylor expansion, we find

$$\varphi(y) = \frac{-\phi(y) + u^{(N)}[x(y)]}{\delta} + \frac{\varepsilon_u^{(N)}[x(y)]}{\delta} + O(\delta), \quad (6.12)$$

where  $x(y)$  is the point of region  $B_1$  which is at a distance  $\delta$  from the point  $y$  along the normal

$$u^{(N)}(x) = -\frac{1}{4\pi} \iint_s \frac{1}{r(x, y)} \varphi^{(N)}(y) ds + F(x),$$

where

$$F(x) = \frac{1}{4\pi} \iint_s \frac{\partial}{\partial n_y} \left[ \frac{1}{r(x, y)} \right] \phi(y) ds.$$

Substituting (6.12) into (6.11), we obtain

$$\begin{aligned} \varepsilon_u^{(N)}(x_0) = & \frac{1}{4\pi} \iint_s \frac{1}{r(x_0, y)} \left[ \frac{-\phi(y) + u^{(N)}[x(y)]}{\delta} + \frac{\varepsilon_u^{(N)}[x(y)]}{\delta} + \right. \\ & \left. + O(\delta) - \varphi^{(N)}(y) \right] ds_y. \end{aligned} \quad (6.13)$$

It is easy to see that for a wide class of surfaces (for example, if  $s \in \Lambda_1$  [36]) and for a sufficiently small  $\delta > 0$ , we can take  $\sigma$  constant and in such case  $x(y) \in B_1$ . Then (6.13) assumes the following form:

$$\begin{aligned} \sigma \varepsilon_u^{(N)}(x_0) = & \frac{1}{4\pi} \left[ \iint_s \frac{u^{(N)}[x(y)] - \phi(y)}{r(x_0, y)} ds_y + 4\pi \sigma u^{(N)}(x_0) - \right. \\ & \left. - \sigma F(x_0) + \iint_s \frac{\varepsilon_u^{(N)}[x(y)]}{r(x_0, y)} ds + O(\delta^2) \right]. \end{aligned} \quad (6.14)$$

Let us analyze the space  $\mathfrak{M}$  of all integrable functions  $\omega(x)$ , determined in the region  $B_0$  with the boundary  $s_0$ . The norm of the element of this space is introduced as

$$\left\| \omega(x \in B_0) \right\|_{\mathfrak{M}} = \left| \iint_{s_0} \frac{\omega(y)}{r(x_0, y)} ds_y \right|,$$

where  $\omega(y) = \omega(x)|_s$ ,  $x_0$  is a certain strictly interior point of the region  $B_1$ .

From (6.14) it is easy to obtain an approximate computation of the error  $\varepsilon_u^{(N)}(x)$  from the norm of the space  $\mathfrak{M}$ . In fact we have

$$\left\| \varepsilon_u^{(N)}(x \in B) \right\|_{\mathfrak{M}} = \left| \iint_s \frac{\varepsilon_u^{(N)}[x(y)]}{r(x_0, y)} ds_y \right| < \left| \iint_s \frac{u^{(N)}[x(y)] - \psi(y)}{r(x_0, y)} ds_y \right| + \\ + O(\delta) + O(\delta^2),$$

where

$$O(\delta) = \delta [ |\varepsilon_u^{(N)}(x_0)| + 4\pi |u^{(N)}(x_0)| + |F(x_0)| ] \\ O(\delta^2) = \delta^2 \frac{M_1}{2},$$

$M_1$  is the maximum of the absolute value of the first derivative of the unknown function  $\varphi$  in the closed region  $\bar{B}$ . After taking  $\delta$  sufficiently small, we find the approximate value

$$\left\| \varepsilon_u^{(N)}(x \in B) \right\|_{\mathfrak{M}} < \left| \iint_s \frac{u^{(N)}[x(y)] - \psi(y)}{r(x_0, y)} ds_y \right|. \quad (6.15)$$

The norm of the space  $\mathfrak{M}$  does not give the possibility for computing the maximum of the modulus of error  $\varepsilon_u^{(N)}$ . If the right-hand side of the computation (6.15) is small, then this still does not indicate smallness of  $\varepsilon_u^{(N)}$ , but, if the right-hand side of Expression (6.15) is large then the error  $\varepsilon_u^{(N)}$ , at least near the boundary, will be sufficiently large. This is the meaning of Expression (6.15).

It is also easy to obtain an approximate a posteriori computation for the error  $\varepsilon_u^{(N)}(y)$  in the sense of the metric of the space  $C^u$ . In fact, if we take (6.12) into account, we find

$$\varepsilon_{\psi}^{(N)}(y) = \frac{u^{(N)}[x(y)] - \psi(y)}{\delta} - \varphi^{(N)}(y) + \frac{\varepsilon_u^{(N)}[x(y)]}{\delta} + O(\delta) \quad /50$$

or

$$|\varepsilon_u^{(N)}[x(y)]| < |u^{(N)}[x(y)] - \psi(y)| + O(\delta) + O(\delta^2),$$

where

$$\begin{aligned} \varepsilon_{\psi}^{(N)}(y) &= \varphi(y) - \varphi^{(N)}(y) \\ O(\delta) &= \varepsilon_{\psi}^{(N)}(y) \delta, \\ O(\delta^2) &= \delta^2 \frac{M_1}{2}, \end{aligned}$$

The order "0" we obtain from the convergence of the first procedure, i.e., from the condition  $\varepsilon_{\psi}^{(N)}(y) \rightarrow 0$ . With a sufficiently small  $\delta$  we find the following simple approximate computation:

$$|\varepsilon_u^{(N)}[x(y)]| < |u^{(N)}[x(y)] - \psi(y)|.$$

The estimate of (6.16) is derived for the points  $x \in B_i$ , distributed on the surface  $\bar{S}$ , which remains at a distance  $\delta$  from the boundary  $s$ .

Let us denote the region bounded by the surfaces  $S$  and  $\bar{S}$  as  $B'$ . From (6.11) it is clear that  $u^{(N)}(x)$  is a harmonic function, and therefore, on the strength of the maximum principle, we obtain

$$\max_{x \in B_i - B'} |\varepsilon_u^{(N)}(x)| < \max_{x \in \bar{S}} |u^{(N)}[x(y)] - \psi(y)|. \quad (6.16)$$

Let us note that for the points distributed at a sufficient distance from the boundary  $s$ , computation of (6.16) may give overly large values. Passing to the limit when  $\delta \rightarrow 0$ , the approximate computations of (6.14) - (6.16) become completely strict. However, for their use we must compute the integrals of the unbounded functions [37].

## §7. Series of Nonorthogonal Systems of Functions

Many problems of applied mathematics are reduced to obtaining expansion of the functions  $\varphi(x) \in L_2(G)$ , where  $G$  is the region for determining the variable  $x$ , in a series of functions of the complete system  $\{\varphi_i(x)\}$ . For convergence in  $L_2(G)$  of the series

$$\sum_{i=1}^N a_i^{(N)} \varphi_i(x) \quad (7.1)$$

to  $\varphi(x)$  it is sufficient to obtain, from the given system, a linearly independent system (after excluding the "extra" functions), then the orthonormalized system  $\{\omega_i(x)\}$

$$\omega_i(x) = \sum_{k=1}^i A_{h,i} \varphi_k(x)$$

and to take the Fourier series of functions  $\varphi(x)$  according to the system  $\{\omega_i(x)\}$ ,

$$\sum_{i=1}^N b_i \omega_i(x), \quad b_i = \int_G \varphi(x) \omega_i(x) dx. \quad /51$$

For the coefficients  $a_i^{(N)}$  of Series (7.1) we obtain

$$a_i^{(N)} = \sum_{k=i}^N A_{h,i} b_k.$$

In practice, orthonormalization of a large<sup>(8)</sup> number of linearly independent functions involves significant difficulties. First with orthonormalization of  $n$ -functions we are required to compute with a high degree of accuracy the elements of the Gram determinant, i.e., if we take into account the symmetry of this determinant, and  $n(n+1)/2$  integrals of the types

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<sup>(8)</sup> On the numerical examples [22, 23], carried out with the aid of standard programs [27, 28], we were convinced that in a number of cases the method of the Fourier series with a low degree of accuracy (2 digits) requires a small number of expansion terms. However, a further increase in accuracy significantly increases the required number of expansion terms, and therefore a large number of functions must be orthonormalized.

$$\int_G \varphi_i(x) \varphi_j(x) dx.$$

This requires a large amount of machine time. Thus, for example [22, 23], orthonormalization of the system

$$\{\ln r(M_i, M)\} \quad (i=1, 2, 3, \dots, 28), \quad (7.2)$$

where  $r(M_i, m)$  is the distance between the fixed point  $M_i$  and the variable  $M$ ,  $M_i \in s_1$ ,  $M \in s$ ,  $s$  and  $s_1$  are concentric circles with radii of 1 and 1.05 requires about two and a half hours of machine time on the BESM-2 (the accuracy of computing the integrals is  $10^{-6}$ ).

The second difficulty is more substantial and involves the smallness of the Gram determinant for unreliable linearly independent systems [16, 24] (which are not a Barry basis [38]). Thus, for the unreliable [22, 23] System (7.2) in the case of concentric circles  $S$  and  $S_1$  with radii 1 and 2 (the points  $M_i$  were distributed uniformly on the circle  $s_1$ ) we were unable [22] from the respective Gram determinant to obtain a tenth-order determinant that is nonmachine zero (computations were carried out on the BESM-2 computer), although we know [1] that the system (7.2) is linearly independent. Thus, to orthonormalize ten functions from System (7.2) on the BESM-2 (39 digits, of which 32 are for the mantissa) is impossible. We must also bear in mind that the ordinary summation of the Fourier series is not stable, and the problem of approximate determination of the functions at a certain point according to the approximate values of the Fourier coefficients in the metric function  $l_2$  is an improper problem [39].

From the above it becomes clear that it is important to construct algorithms for obtaining the coefficients  $a_i^{(N)}$  of series (7.1) with respect to /52 nonorthogonal systems of functions. It is clear that taking the Fourier coefficients as  $a_i^{(N)}$  in (7.1) generally speaking, will not guarantee convergence. Thus, we know [1, 3] that if the points  $M_i$  are distributed everywhere densely on  $S_1$ , then System (7.2) is complete in  $L_2(s)$ . However, if we expand the constant  $C$  over  $S$  in a Fourier series according to System (7.1), we obtain

$$C \sim \sum_{i=1}^N a_i^{(N)} \ln r(M_i, M) = a \sum_{i=1}^N \ln r(M_i, M),$$

where

$$a = C \int_S \ln r(M_i, M) ds_M.$$

(the integral in the right-hand side of the last expression does not depend on the location of the point  $M_i$  on  $S_1$ ).

In the present section we discuss a new method [25] for expanding the functions in a series of nonorthogonal systems of functions. In the theorem discussed below we give sufficient conditions for convergence of the respective series. As will be shown below, the proposed series give a slightly better approximation in comparison with the Fourier series for the approximately orthonormalized functions.

Let us seek the expansion  $\sum_i a_i \varphi_i$  of the function  $\varphi(x)$  with respect to the system  $\{\varphi_i(x)\}$  ( $i=1, 2, \dots$ ), the functions of which will be assumed normalized. The essence of the proposed method is as follows:

The first coefficient  $a_1$  coincides with the Fourier coefficient for the functions

$$a_1 = \int_G \varphi \varphi_1 dx.$$

The difference  $\varphi - a_1 \varphi_1$  is termed the first remainder. The second coefficient is the Fourier coefficient for the first remainder

$$a_2 = \int_G (\varphi - a_1 \varphi_1) \varphi_2 dx.$$

The difference  $\varphi - \sum_{i=1}^k a_i \varphi_i$  is termed the  $k^{\text{th}}$  remainder. The  $(k+1)^{\text{th}}$  coefficient is the Fourier coefficient for the  $k^{\text{th}}$  remainder

$$a_{k+1} = \int_G \left( \varphi - \sum_{i=1}^k a_i \varphi_i \right) \varphi_{k+1} dx.$$

Let us introduce the following definitions:

$$\varphi^{(n)}(x) = - \sum_{i=0}^n a_i \varphi_i(x), \quad \varphi_0(x) = \varphi(x), \quad a_0 = -1, \quad (7.3)$$

$$a_i = \int_G \varphi^{(i-1)}(x) \varphi_i(x) dx \quad (i=1, 2, \dots), \quad \varphi^{(0)}(x) = \varphi(x). \quad (7.4)$$

The proof that  $\|\varphi^{(n)}(x)\|_{L_2} \rightarrow 0$  when  $n \rightarrow \infty$  will be equivalent to proof of the convergence of the Series (7.1) in the sense of the metric  $L_2(G)$ .

It is obvious from Formulas (7.3), (7.4) that with an increase in the /53  
number of expansion terms in the Series (7.1) the previous coefficients do not change, and we must compute only the coefficients for the new expansion terms. This significant advantage (from the computational view point) is simultaneously a substantial disadvantage of the proposed method. In fact in many cases it is certainly known that the function  $\varphi(x)$  is not represented in the form of an infinite series  $\sum_{k=1}^{\infty} a_k \varphi_k(x)$ , and therefore, in these cases the proposed method of constructing the series will not be convergent.

The sequence of positive numbers  $\|\varphi^{(n)}(x)\|_{L_2} = \sqrt{(\varphi^{(n)}(x) \varphi^{(n)}(x))} \quad (n=1, 2, \dots)$  is monotonically decreasing and consequently has a limit which we can denote by R. In fact [10],

$$\begin{aligned} \|\varphi^{(n)}(x)\|_{L_2}^2 &= \min_{a_n} \left\| \varphi^{(n-1)}(x) - a_n \varphi_n(x) \right\|_{L_2}^2 = \frac{G(\varphi^{(n-1)} \varphi_n)}{G(\varphi_n)} = \\ &= \left\| \varphi^{(n-1)}(x) \right\|_{L_2}^2 - \left[ \int_G \varphi^{(n-1)}(x) \varphi_n(x) dx \right]^2, \end{aligned}$$

where  $G(\varphi_1, \varphi_2, \dots, \varphi_n)$  is the Gram determinant of the function  $\varphi_1, \varphi_2, \dots, \varphi_n$ .

We can also show that for any  $\varepsilon > 0$  we find a finite  $N_0$  such that the following inequality will be satisfied

$$\sum_{k=N_0}^{\infty} (\varphi^{(k)}, \varphi_{k-1})^2 < \varepsilon,$$

where  $(\varphi, \psi)$  is the scalar product of the functions  $\varphi$  and  $\psi$ . In fact, for any  $\varepsilon > 0$ , we find an  $N_0$  such that

$$\varepsilon > \left\| \varphi^{(N_0)}(x) \right\|_{L_2}^2 - R = \sum_{k=N_0}^{\infty} (\varphi^{(k)}, \varphi_{k+1})^2. \quad (7.5)$$

Taking into account the notation of (7.3) and (7.4) and the normalization of the system  $\{\varphi_i(x)\}$ , for any  $\varepsilon > 0$  we find an  $N_0$  such that when  $S > N_0$  we will have

$$\begin{aligned} \varepsilon > \left| \left\| \varphi^{(s)}(x) \right\|_{L_0}^2 - \left\| \varphi^{(s+1)}(x) \right\|_{L_2}^2 \right| &= \left| \int_G \left[ \sum_{i=0}^s a_i \varphi_i(x) \right]^2 dx - \right. \\ &- \int_G \left[ \sum_{i=0}^{s+1} a_i \varphi_i(x) \right]^2 dx \left| = \left| -2a_{s+1} \int_G \varphi^{(s)}(x) \varphi_{s+1}(x) dx - \right. \right. \\ &\left. \left. - a_{s+1}^2 \int_G \varphi_{s+1}^2(x) dx \right| = a_{s+1}^2. \end{aligned}$$

Thus, we find that for any  $\varepsilon > 0$  and a whole (finite)  $N$  we find an  $N_0$  such that

$$\varphi^{(s)}(x) = \varphi^{(r)}(x) + \gamma_r^{(s)}(x) \quad (N_0 < S, \quad r < N_0 + N), \quad (7.6) \quad \underline{/54}$$

where

$$\left\| \gamma_r^{(s)}(x) \right\|_{L_2} < \varepsilon_1. \quad (7.7)$$

Substituting (7.6) into (7.5) we find

$$\begin{aligned} \varepsilon > \sum_{k=N_0}^{\infty} (\varphi^{(k)}, \varphi_{k+1})^2 &> \sum_{k=N_0}^{N_0+N} (\varphi^{(k)}, \varphi_{k+1})^2 = \sum_{k=N_0}^{N_0+N} (\varphi^{(r)}, \varphi_{k+1})^2 + \\ &+ \sum_{k=N_0}^{N_0+N} (\gamma_r^{(k)}, \varphi_{k+1})^2 + 2 \sum_{k=N_0}^{N_0+N} (\varphi^{(r)}, \varphi_{k+1}) (\gamma_r^{(k)}, \varphi_{k+1}). \end{aligned} \quad (7.8)$$

Using the Buniakowski-Schwarz inequality and taking (7.7) into account, the second and third terms in the right-hand side of Expression (7.8) for any finite  $N$  can be made as small as desired and therefore, we ultimately find that for any  $\varepsilon_2 > 0$  and  $N$  we find an  $N_0$  such that

$$\sum_{k=N_0}^{N_0+N} (\varphi^{(r)}, \varphi_{k+1})^2 < \varepsilon_2 \quad (N_0 \leq r < N_0 + N). \quad (7.9)$$

Thus, the difference between the expanded function  $\varphi(x)$  and its series  $\sum_{i=1}^r a_i \varphi_i(x)$  is "almost" orthogonal to as large a number of functions of the system  $\{\varphi_i(x)\}$  as desired.

We shall assume that the function  $\varphi(x)$  and the system  $\{\varphi_i(x) | i = 1, 2, \dots\}$  satisfy the following conditions: for any  $\varepsilon > 0$  and  $N_0$  we find coefficients  $b_k$  ( $k = N_0, N_0 + 1, \dots, N_0 + N$ ), such that

$$\left\| \varphi^{(r)}(x) - \sum_{k=N_0}^{N_0+N} b_k \varphi_k(x) \right\|_{L_2} < \varepsilon, \quad (7.10)$$

$$\sum_{k=N_0}^{N_0+N} b_k^2 < M, \quad (7.11)$$

where  $M$  is independent of  $N_0$  and  $N$  is constant, and  $r$  is any whole number satisfying the inequalities  $N_0 \leq r < N_0 + N$ .

According to the familiar Muntz theorem [10], Condition (7.10) for a fixed  $r$  is satisfied by the system  $\{x^{k_i}\}$  ( $i = 1, 2, \dots$ ) when  $k_i > -\frac{1}{2}$ ,  $\lim_{i \rightarrow \infty} k_i = \infty$ ,  $\sum_{i=1}^{\infty} \frac{1}{k_i} = \infty$ , where the prime denotes omission of possible  $k_1 = 0$ . The same condition for a fixed  $r$  is satisfied by the so-called potential system [1,23]. In other words, both the potential systems and the system  $\{x^{k_i}\}$  possess the property that, after eliminating any finite number of functions, they again become complete in  $L_2$ . We must however, mention that in inequality (7.10)  $r$  depends 55 on  $N$ . In fact, after increasing  $N$ , as follows from (7.9) we must increase  $N_0$ . Therefore, we can never state that for these systems inequality (7.10) is satisfied. Satisfaction of inequality (7.10) depends both on the system  $\{\varphi_i(x)\}$ , and on the expanded function  $\varphi(x)$ .

Condition (7.11) when  $N_0 = 1$  is satisfied by the Fourier coefficients according to the complete orthonormalized system (the equation of closure).

The following theorem is valid. The series  $\sum_{i=1}^N a_i \varphi_i(x)$ , where the  $a_i$  are computed from (7.4), of functions  $\phi(x)$  with respect to the system  $\{\phi_i(x)\}$ , which satisfy Conditions (7.10) and (7.11), converges when  $N \rightarrow \infty$  to this function in the sense of the metric of the space  $L_2(G)$ .

In fact from (7.9), (7.11) and the Buniakowski-Schwarz inequality we obtain

$$\sum_{k=N_0}^{N_0+N} b_k(\varphi^{(r)}, \varphi_{k+1}) < \sqrt{M} \varepsilon < \varepsilon_3, \quad (7.12)$$

where the  $b_k$  satisfy the condition

$$\left\| \varphi^{(r)}(x) - \sum_{k=N_0}^{N_0+N} b_k \varphi_k(x) \right\| < \varepsilon_4, \quad (7.13)$$

$\varepsilon_3$  and  $\varepsilon_4$  are numbers as small as desired.

From (7.12) and (7.13) we obtain

$$(\varphi^{(r)}, \varphi^{(r)}) = \sqrt{\|\varphi^{(r)}\|_{L_2}^2} < \varepsilon_5,$$

where  $\varepsilon_5$  is a number as small as desired.

Condition (7.10) of the proven theorem contains the function  $\phi^{(r)}(x)$  and consequently it is difficult to prove. Therefore, the following assumption is of interest. If the function  $\phi(x)$  and the system  $\{\phi_i(x)\}$  ( $i = 1, 2, \dots$ ) satisfy the conditions: for any whole finite  $N_0$  and any  $u(x) \in L_2$  we find coefficients  $b_k$  ( $k = N_0, N_0 + 1, \dots$ ), such that

$$\lim_{n \rightarrow \infty} \left\| u(x) - \sum_{k=N_0}^n b_k \varphi_k(x) \right\|_{L_2} = 0, \quad (7.10_1)$$

$$\sum_{k=N_0}^{\infty} b_k^2 < M, \quad (7.11_1)$$

where  $M$  is a constant independent of  $N_0$ ; for any  $\varepsilon > 0$  we find an  $n_0$  such that

$$\sum_{k=n_0+1}^{\infty} A_k < \varepsilon, \quad (7.14)$$

where

$$A_k = \sum_{i=n_0+1}^k a_i(\varphi_i, \varphi_{k+1}) (2\varphi^{(n_0)} + \sum_{j=n_0+1}^k a_j(\varphi_j, \varphi_{k+1})),$$

then the series

/56

$$\sum_{j=1}^n a_j \varphi_j(x),$$

where  $a_j$  are computed from (7.4), converges on the average when  $n \rightarrow \infty$  to the function  $\varphi(x)$ .

In fact from (7.5) we find that for any  $\varepsilon > 0$  we can find an  $n_0$  such that

$$\begin{aligned} \varepsilon &> \sum_{k=n_0}^{\infty} (\varphi^{(k)}, \varphi_{k+1})^2 = \sum_{k=n_0}^{\infty} (\varphi^{(n_0)}, \varphi_{k+1})^2 + \sum_{k=n_0}^{\infty} (\varphi^{(k)} - \varphi^{(n_0)}, \varphi_{k+1})^2 + \\ &+ 2 \sum_{k=n_0}^{\infty} (\varphi^{(n_0)}, \varphi_{k+1}) (\varphi^{(k)} - \varphi^{(n_0)}, \varphi_{k+1}) = \sum_{k=n_0}^{\infty} (\varphi^{(n_0)}, \varphi_{k+1})^2 + \\ &+ \sum_{k=n_0+1}^{\infty} \left( \sum_{i=n_0+1}^k a_i \varphi_i, \varphi_{k+1} \right)^2 + 2 \sum_{k=n_0+1}^{\infty} \left( \varphi^{(n_0)}, \varphi_{k+1} \right) \left( \sum_{i=n_0+1}^k a_i \varphi_i, \varphi_{k+1} \right). \end{aligned}$$

From (7.14) and the last inequality we find that for any  $\varepsilon > 0$  we find an  $n_0$  such that

$$\sum_{k=n_0}^{\infty} (\varphi^{(n_0)}, \varphi_{k+1}) < \varepsilon.$$

Using (7.10<sub>1</sub>) and (7.11<sub>1</sub>) for the function  $\varphi^{(n_0)} \in L_2$  when  $N_0 = n_0$  we obtain

$$\lim_{n \rightarrow \infty} \left\| \varphi^{(n_0)}(x) - \sum_{k=n_0}^n b_k \varphi_k(x) \right\|_{L_2} = 0,$$

$$\sum_{k=n_0}^{\infty} b_k^2 < M$$

or using the Buniakowski-Schwarz inequality and the theorem<sup>(9)</sup> on continuity of the scalar product

$$\sum_{k=n_0}^{\infty} b_k (\varphi^{(n_0)}, \varphi_{k+1}) = (\varphi^{(n_0)}, \varphi^{(n_0)}) = \left\| \varphi^{(n_0)} \right\|_{L_2}^2 < \sqrt{M} \varepsilon < \bar{\varepsilon},$$

where  $\bar{\varepsilon}$  is a quantity that is as small as desired.

Condition (7.10<sub>1</sub>), unlike Condition (7.10), has a simple meaning: the system  $\{\varphi_i(x)\}$  ( $i = N_0, N_0 + 1, \dots$ ), where  $N_0$  is any finite whole number, must remain complete in  $L_2$ . Let us prove that the system  $\{\ln r(M_i, M)\}$  of discontinuous potential functions [22, 24] where  $M \in S$  and  $M_i \in S$  satisfies Conditions (7.10<sub>1</sub>) and (7.11<sub>1</sub>) for the space  $L_2(s)$ . Let the function  $\gamma(M_i) \in L_2(s)$  be orthogonal to all functions of this system. We must prove that  $\gamma(M) = 0$ . Let us analyze the integral operator

/57

$$\int_s \ln r(M_1, M) \gamma(M_1) ds_{M_1} = \Phi(M)$$

which transforms the space  $L_2(s)$  ( $\gamma(M_i) \in L_2(s)$ ) into the space  $\text{Lip } \alpha$  ( $\alpha < \frac{1}{2}$ ), ( $\Phi(M) \in \text{Lip } \alpha$ ). The continuous function  $\Phi(M)$  takes zero values on the set of points  $M_1$  that is everywhere dense on  $S$  (in view of the orthogonality of  $\gamma(M_i)$  to all functions of the examined system), and consequently it is equal to absolute zero on  $s$ , and on the strength of its harmonicity, everywhere outside  $s$  also. But, the potential of the single layer  $\int_s \ln r(M_1, M) \gamma(M_1) ds_{M_1}$  is then identically equal to zero, when its density  $\gamma(M_1)$  identically equals zero. Let us now show that for any  $\varepsilon > 0$ ,  $u(M) \in L_2(s)$  and  $N_0$  we find coefficients  $b_k$  ( $k = N_0, N_0 + 1, \dots, N_0 + N$ ), such that

$$\left\| u(M) - \sum_{k=N_0}^{N_0+N} b_k \ln r(M_k, M) \right\|_{L_2(s)} < \varepsilon.$$

---

(9) We bear in mind the known theorem that if  $\varphi_n \rightarrow \varphi$  and  $\psi_n \rightarrow \psi$ , then  $(\varphi_n, \psi_n) \rightarrow (\varphi, \psi)$ .

On the strength of the proven completeness of the system being analyzed, we find coefficients  $C_k$  ( $k = 1, 2, \dots, N_1$ ), such that

$$\left\| u(M) - \sum_{j=1}^{N_1} C_j \ln r(M_j, M) \right\|_{L_2(s)} < \frac{\varepsilon}{2}.$$

Since the points  $M_k$  are distributed everywhere densely on  $s$ , then from any point  $M_j$  ( $j = 1, 2, \dots, N_1$ ) we find a point  $M_{r_j}$  such that  $r_j > N_0$  and

$$\left\| \ln r(M_j, M) - \ln r(M_{r_j}, M) \right\|_{L_2(s)} < \frac{\varepsilon}{2N_1 C},$$

where  $C = \max_j |C_j|$ . Taking this latter inequality and the Minkowski inequality into account, we find

$$\begin{aligned} \left\| u(M) - \sum_{k=N_0}^{N_0+N} b_k \ln r(M_k, M) \right\|_{L_2(s)} &< \left\| u(M) - \sum_{j=1}^{N_1} C_j \ln r(M_j, M) \right\|_{L_2(s)} + \\ &+ \left\| \sum_{j=1}^{N_1} C_j \ln r(M_j, M) - \sum_{k=N_0}^{N_0+N} b_k \ln r(M_k, M) \right\|_{L_2(s)}. \end{aligned}$$

After taking for  $N$  and  $b_k$  ( $k = N_0, \dots, N_0 + N$ ) the following values:

$$N = \max_j r_j - N_0, \quad b_k = \begin{cases} C_j & \text{when } M_k = M_{r_j} \\ 0 & \text{when } M_k \neq M_{r_j} \end{cases}$$

we find

/58

$$\begin{aligned} \left\| u(M) - \sum_{k=N_0}^{N_0+N} b_k \ln r(M_k, M) \right\| &< \frac{\varepsilon}{2} + \left\| \sum_{j=1}^{N_1} C_j [\ln r(M_j, M) - \right. \\ &\quad \left. - \ln r(M_{r_j}, M)] \right\|_{L_2(s)} < \varepsilon. \end{aligned}$$

Now let us proceed to proving (7.11<sub>1</sub>). For the system  $\{\ln r(M_k, M)\}$  we can prove a significantly stronger assumption: for any function  $u(M) \in L_2(s)$ , any  $N_0$  and any  $\varepsilon > 0$ , we find such coefficients  $b_k$  ( $k = N_0, \dots, N_0 + N$ ), that

$$\left\| u(M) - \sum_{k=N_0}^{N_0+N} b_k \ln r(M_k, M) \right\|_{L_2(s)} < \varepsilon \quad (7.10_2)$$

and

$$\sum_{k=N_0}^{N_0+N} b_k^2 < \varepsilon. \quad (7.11_2)$$

Let us represent  $u(M)$  in the form of a potential of the single layer

$$u(M) = \int_s v(\bar{M}) \ln r(M, \bar{M}) ds_{\bar{M}},$$

and substitute the integral in the right-hand side by the Riemann sum and prove that for any  $\varepsilon > 0$  we find an  $N$  such that

$$\Delta = \int_s \left[ \int_s v(\bar{M}) \ln r(M, \bar{M}) ds_{\bar{M}} - \sum_{k=1}^N h_k v(M_k) \ln r(M, M_k) \right]^2 ds_{\bar{M}} < \varepsilon,$$

where  $h_k$  is the length of the  $k^{\text{th}}$  segment dividing the boundary  $s$ , in which we take the point  $M_k$ . Taking into account that the points  $M_k$  are distributed everywhere densely on  $s$ , and  $v(s)$  is a continuous function, we obtain

$$\begin{aligned} \Delta &= \int_s \left[ \int_s v(\bar{M}) \ln r(M, \bar{M}) ds_{\bar{M}} \right]^2 ds_{\bar{M}} - \\ &- 2 \sum_{k=1}^N h_k v(M_k) \int_s \ln r(M, M_k) \int_s v(\bar{M}) \ln r(M, \bar{M}) ds_{\bar{M}} ds_{\bar{M}} + \\ &+ \sum_{k=1}^N \sum_{j=1}^N h_k v(M_k) h_j v(M_j) \int_s \ln r(M, M_k) \ln r(M, M_j) ds_{\bar{M}} = \\ &= \int_s \left[ \int_s v(\bar{M}) \ln r(M, \bar{M}) ds_{\bar{M}} \right]^2 ds_{\bar{M}} - \end{aligned}$$

/59

$$\begin{aligned} &- 2 \int_s v(\bar{M}) \int_s \ln r(M, \bar{M}) \int_s v(\bar{M}) \ln r(M, \bar{M}) ds_{\bar{M}} ds_{\bar{M}} ds_{\bar{M}} + \\ &+ \varepsilon_1 + \iint_{s \ s} v(\bar{M}) v(\tilde{M}) \int_s \ln r(M, \bar{M}) \ln r(M, \tilde{M}) ds_{\bar{M}} ds_{\tilde{M}} ds_{\bar{M}} + \varepsilon_2 = \varepsilon_1 + \varepsilon_2, \end{aligned}$$

where  $\varepsilon_1$  and  $\varepsilon_2$  are numbers as small as desired. Thus, as the coefficients  $b_k$  we can take the numbers  $h_k v(M_k)$  and therefore,

$$\sum_{k=N_0}^{N_0+N} b^2 = \sum_{k=N_0}^{N_0+N} h_k^2 v^2(M_k) < h^{(N)} \sum_{k=N_0}^{N_0+N} h_k v^2(M_k) = h^{(N)} \left[ \int_s v^2(M) ds + \varepsilon_3 \right]$$

where  $\varepsilon_3$  is a number as small as desired and  $h^{(N)} = \max_{k \leq N} h_k$ . Since when  $N \rightarrow \infty$   $h^{(N)} \rightarrow 0$ , then from the latter expression there directly follows (7.11<sub>2</sub>). Satisfaction of Conditions (7.10<sub>2</sub>), (7.11<sub>2</sub>) makes it possible to prove the following assumption for the system  $\{\ln r(M_k, M)\}$ . If for any  $\varepsilon > 0$  we find such an  $N$  that

$$h^{(N)} \sum_{k=n_0+1}^N A_k < \varepsilon, \quad (7.14_1)$$

then the series

$$\sum_{j=1}^n a_j \ln r(M_j, M),$$

where  $a_j$  are computed from (7.4), converges on the average when  $n \rightarrow \infty$  to the function  $\varphi(x)$ . In fact, from Condition (7.14<sub>1</sub>) we find

$$\bar{C} V \varepsilon > \bar{C} h^{(N)} \sum_{k=n_0+1}^N A_k > \sum_{k=n_0+1}^N b_k (\varphi^{(n_0)}, \varphi_{k+1}) = \left\| \varphi^{(n_0)} \right\|_{L_2}^2 + \varepsilon,$$

where  $\tilde{\varepsilon}$  is a quantity as small as desired and

$$\tilde{c} = \sqrt{\int_s v^2(s) ds}.$$

Thus, we find that the norm of the functions  $\varphi^{(n_0)}$  can be made as small as desired.

Conditions (7.10), (7.11) are sufficient but not necessary conditions for convergence of the proposed series. Thus, for complete orthonormalized systems, Condition (7.10) is not satisfied. However, for these systems the

given series (in this case they coincide with the Fourier series) converge. Let us also mention that the difficulties in orthonormalization arise namely for systems which satisfy Condition (7.10) (not minimal systems [40]), and therefore, for such systems construction of these series has an advantage over the Fourier series (in the sense of practical application).

A trivial example of the system which satisfies Conditions (7.10), (7.11) /60 can be constructed in the following manner. Let  $P_1, P_2, \dots$  be an arbitrary infinite sequence of increasing whole numbers,  $\lim_{i \rightarrow \infty} P_i = \infty$ . From the orthonormalized complete system  $\{\varphi_i(x)\}$  we can formulate the following normalized system  $\{\psi_i(x)\}$ , where  $\psi_i(x) = \varphi_j(x)$ ,  $j = i - \sum_{k=1}^r P_k$ ,  $r$  is the greatest whole number for which  $\sum_{k=1}^r P_k < i$ .

The system  $\{\psi_i(x)\}$  has the following form:

$$\{\varphi_1, \varphi_2, \dots, \varphi_{P_1}, \varphi_1, \varphi_2, \dots, \varphi_{P_2}, \varphi_1, \varphi_2, \dots, \varphi_{P_3}, \varphi_1, \varphi_2, \dots\}. \quad (7.15)$$

It is easy to see that when  $\lim_{i \rightarrow \infty} P_i = \infty$ , it satisfies Conditions (7.10), and (7.11). Let us analyze the system  $\{\psi_i\}$

$$\{\varphi_1, \varphi_2, \dots, \varphi_{P_1}, \varphi_{n_1}, \varphi_{n_1+1}, \dots, \varphi_{P_1}, \varphi_{n_2}, \varphi_{n_2+1}, \dots, \varphi_{P_2}, \dots\}, \quad (7.16)$$

where  $n_i, P_i$  are whole numbers, and the system  $\{\phi_i\}$  is orthonormalized. We can show that the coefficient of the series computed from (7.4) for the function  $\Phi_s$  ( $s$  is an arbitrary whole number) of the Series (7.16) coincides with the Fourier coefficient for the function  $\Phi_s$ , if  $\Phi_s$  was first encountered in Series (7.16); and is equal to zero if  $\Phi_s$  prior to this was encountered in Series (7.16). In fact from (7.4) for the first case we have

$$a_s = \int_G \left( \varphi - \sum_{k=1}^{s-1} a_k \Phi_k \right) \Phi_s dx = \int_G \varphi \Phi_s dx,$$

and for the second case ( $\Phi_s \equiv \Phi_r$ , where  $r$  is a whole number smaller than  $s$ )

$$a_s = \int_G \left( \varphi - \sum_{k=1}^{s-1} a_k \Phi_k \right) \Phi_s dx = \int_s \varphi \Phi_s dx - a_r \int_s \Phi_r \Phi_s dx = 0.$$



to one single iteration for System (7.17), when as the initial approximation for the vector  $(\lambda_1, \dots, \lambda_n)$  we take the null vector  $\lambda^{(0)} (0, \dots, 0)$ , i.e.,

$$a_i^{(\Phi)} = \lambda^{(1)} + B \lambda^{(0)} + R,$$

where  $a_i^{(\Phi)}$  is a vector whose components are Fourier coefficients, B is a matrix which corresponds to the following description of System (7.17)

$$\begin{aligned} \lambda &= B \lambda + R, \\ \lambda^{(1)} &= (\lambda_1^{(1)}, \dots, \lambda_n^{(1)}), \lambda^{(0)} = (0, \dots, 0), R = ((\varphi, \varphi_1), \dots, (\varphi, \varphi_n)). \end{aligned} \quad (7.18)$$

Let us write (7.18) in the following form:

$$\lambda = (B_1 + B_2) \lambda + R,$$

where

$$B_1 = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & 0 \\ -(\varphi_1, \varphi_2) & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -(\varphi_1, \varphi_n) & -(\varphi_2, \varphi_n) & \dots & -(\varphi_{n-1}, \varphi_n) & 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & -(\varphi_2, \varphi_1) & -(\varphi_3, \varphi_1) & \dots & -(\varphi_n, \varphi_1) \\ 0 & 0 & -(\varphi_3, \varphi_2) & \dots & -(\varphi_n, \varphi_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Computation of the Coefficients of (7.4) with respect to the system  $\{\varphi_i\}$  corresponds to one iteration according to Seidel's method for System (7.18) or to one single iteration for the system

$$\lambda = (I - B_1)^{-1} B_2 \lambda + (I - B_1)^{-1} R, \quad (7.19)$$

where I is the unit matrix when as the initial approximation we take the null /62 vector  $\lambda^{(0)} = (0, \dots, 0)$ , i.e.,

$$a^{(A)} = \bar{\lambda}_1^{(1)} = (I - B_1)^{-1} B_2 \lambda^{(0)} + (I - B_1)^{-1} R,$$

where  $a^{(A)}$  is a vector whose components are coefficients of the proposed series. Let us mention that all the eigenvalues of the matrix  $(I - B_1)^{-1} B_2$  are less than unity. In fact, since the quadratic form corresponding to the Gram matrix is positive-definite, then Seidel's method for System (7.18) or, what amounts to the same thing, the method of single iteration for System (7.19) converges<sup>(10)</sup> for any initial approximation and right-hand side. For such an approach the above theorem is equivalent to the following statement. For an approximate solution to System (7.17) for sufficiently large  $n$ , only one iteration is sufficient [it is assumed that the system  $\{\phi_i\}$  and the function  $\phi(x)$  satisfy Conditions (7.10) and (7.11)] according to Seidel's method.

Since computation of the elements of the Gram matrix (scalar products), and naturally the process of orthonormalization, are accomplished with a finite number of digits, then it is clear that the matrix corresponding to System (7.17) will not be a unit matrix. It will somehow be "perturbed". The coefficients of the best approximation here will be found from the system

$$(1 - \varepsilon_{jj}) \lambda_j + \sum_{\substack{k=1 \\ k \neq j}}^n \varepsilon_{k,j} \lambda_k = (\varphi, \varphi_j) \quad (j=1, 2, \dots, n), \quad (7.20)$$

where  $\varepsilon_{i,j}$  are small perturbations produced by round-off errors and errors in computing the scalar products (in the case of the  $L_2$  space — by errors in integrating). Naturally here we assume that  $\max_k \sum_{j=1}^n |\varepsilon_{k,j}| < 1$ . It is interesting which series is more feasible to use in such cases — the Fourier series or the proposed series? More precisely, what relationship exists between the numbers  $||\lambda - a^{(\Phi)}||$  and  $||\lambda - a^{(A)}||$ , where the vector  $\lambda$  is an exact solution to System (7.20).

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<sup>(10)</sup> Since the diagonal elements of the Gram matrix of the linearly independent system are positive, then the positive determinancy of the respective quadratic form is not only a sufficient but also a necessary condition for convergence of Seidel's iteration process (E. Reich, see [42]).

Since in this case Seidel's iteration process converges more rapidly [42] than the single iteration process and the norm of the matrix  $\max_k \sum_{j=1}^n |e_{kj}|$  is subordinate to the first norm of the vectors, then it is clear that the first norm [42] of the vector  $(\lambda - a^{(\Phi)})$  will be no less than the first norm of the vector  $(\lambda - a^{(A)})$ , i.e.,

$$\max_i |\lambda_i - a_i^{(\Phi)}| = \|\lambda - a^{(\Phi)}\|_I \geq \|\lambda - a^{(A)}\|_I = \max_i |\lambda_i - a_i^{(A)}|.$$

Let us note that both the method of the Fourier series and the method of /63 the proposed series require carrying out one approximate integration for computing the coefficient of each new expansion term.

$$a_k^{(\Phi)} = \int_G \varphi \varphi_k dx,$$

$$a_k^{(A)} = \int_G \varphi^{(k-1)} \varphi_k dx.$$

The factor  $\varphi^{(k-1)}$  is involved in the computation of  $a_k^{(A)}$ . Therefore it is natural that its values at individual points would be taken for checking accuracy of the expansion since

$$\varphi^{(k-1)}(x) = \varphi(x) - \sum_{i=1}^{k-1} a_i^{(A)} \varphi_i(x).$$

Numerical Example. In a number of cases for several regions  $G$  there are tabulated coefficients of orthonormalization, and the expansion of the function  $\varphi(x)$  must be carried out in the region  $G'$ , in a certain sense near to  $G$ . Thus, for example in [43] there are tables of coefficients of orthonormalization for the harmonic polynomials which are orthonormal on an ellipse. In [43] 230 ellipses were analyzed with various ratios of the semiaxis  $p = b/a$ . The tables are used for solving the internal Dirichlet problem for the analyzed ellipses. Let us solve the Dirichlet problem for such an ellipse which was not analyzed in [43]. In this case it is natural to use the orthonormalized harmonic polynomials for the ellipse analyzed in [43] with the value  $p = \frac{b}{a}$  that is nearest to the given ellipse. Here the question arises as to which series are used?

When  $G$  and  $G'$  are sufficiently close, this question is analogous to the question analyzed above (for approximate orthonormalization). Here we analyze one numerical example. Let us look at System (7.2) for the ellipses  $s$  and  $s_1$ , with semiaxes 1 and 0.5, 2 and 1, respectively.

$$\left\{ \frac{1}{2} \ln \left[ \left( 2 \cos \alpha_i - \cos \alpha \right)^2 + \left( \sin \alpha_i - \frac{1}{2} \sin \alpha \right)^2 \right] \right\} \quad (i = 1, 2, \dots, 12),$$

where

$$\alpha_1 = 90^\circ, \alpha_2 = 270^\circ, \alpha_3 = 0^\circ, \alpha_4 = 180^\circ, \alpha_5 = 225^\circ, \alpha_6 = 45^\circ, \alpha_7 = 315^\circ, \\ \alpha_8 = 135^\circ, \alpha_9 = 330^\circ, \alpha_{10} = 150^\circ, \alpha_{11} = 120^\circ, \alpha_{12} = 300^\circ.$$

Table 4 gives the coefficients of orthonormalization  $A_{k,i}$  of this system for the interval  $0 < \alpha < 2\pi$ . Let us seek the solution to the internal Dirichlet problem

$$\Delta u = 0, \\ u \Big|_s = \operatorname{arctg} \frac{y-2}{x-2} = \psi(\alpha),$$

for the ellipse  $s_1$ , with semiaxes 1 and 0.45. The coefficients of orthonormalization of the System (7.2) on this ellipse will no longer coincide with those above, since it takes the following form:

$$\left\{ \frac{1}{2} \ln \left[ (2 \cos \alpha_i - \cos \alpha)^2 + (\sin \alpha_i - 0.45 \sin \alpha)^2 \right] \right\} = \{ \varphi_i \}$$

However, the functions

$$\{ \psi_i \} = \left\{ \sum_{k=1}^i A_{k,i} \varphi_k \right\}$$

will be roughly orthonormalized as Table 5 shows in which the Gram matrix is given  $\{(\psi_i, \psi_j)\}$  for the system  $\{\psi_i\}$ . Table 6 gives the Fourier coefficients  $a_i^{(\psi)}$  of the function  $\psi(s)$  for the system  $\{\psi_i\}$ , the coefficients (7.4) of the proposed series  $a_i$  of the function  $\psi(s)$  for the system  $\{\psi_i\}$  and the

coefficients  $\lambda$  of the best approximation in the sense of the metric  $L_2$  of the function  $\psi(s)$  as functions of the system  $\{\psi_i\}$ .  $\lambda_i$  can be found either from the system

$$\sum_{i=1}^n (\psi_i, \psi_j) \lambda_i = (\psi, \psi_j) \quad (j=1, 2, \dots), \quad (7.21)$$

or an orthonormalized system can be obtained on  $s$  for  $\omega_i = \sum_{k=1}^i C_{k,i} \varphi_k$ , the Fourier series of the function  $\psi$  for the system  $\{\omega_i\}$  can be taken

$$\sum_{i=1}^n b_i \omega_i$$

and then we can use the equations

$$\lambda_i = \sum_{k=i}^n C_{k,i} b_k. \quad (7.22)$$

It is clear that both methods must give coefficients of the best approximation and on the strength of the uniqueness of the generalized polynomial of the best approximation for the strictly normalized space  $L_2$ , the two systems obtained by these methods for the coefficients must coincide. Table 6 shows the coefficients  $\lambda_i$ , computed by the second method from (7.22). A few words should be said about computation of the coefficients  $a_i$ . We know [4.2] that Seidel's method gives the greatest advantage in rate of convergence in comparison with ordinary iteration in the case when the equations are arranged in order of increasing  $\sum_j a_{j,i}$ , by taking for the first equation that in which this sum is the smallest. Therefore [see Table 5 which gives the coefficients of System (7.21)] the coefficients  $a_i$  were determined in the following sequence:

$$a_9, a_7, a_4, a_8, a_{10}, a_{11}, a_{12}, a_1, a_2, a_5, a_3, a_6.$$

From the table it is clear that both the first and the second and third norms [42] of the vector  $\lambda - a$  are less than the respective norms of the vector

$$\begin{aligned}\|\lambda - a\|_I &= \max_k |\lambda_k - a_k| = 0,45 < 1,31 = \max_k |\lambda_k - a_k^{(\Phi)}| = \|\lambda - a^{(\Phi)}\|_I \\ \|\lambda - a\|_{II} &= \sum_{k=1}^{12} |\lambda_k - a_k| = 2,3 < 5,9 = \sum_{k=1}^{12} |\lambda_k - a_k^{(\Phi)}| = \|\lambda - a^{(\Phi)}\|_{II} \\ \|\lambda - a\|_{III} &= \sum_{k=1}^{12} (\lambda_k - a_k)^2 = 0,63 < 5,6 = \sum_{k=1}^{12} (\lambda_k - a_k^{(\Phi)})^2 = \|\lambda - a^{(\Phi)}\|_{III}\end{aligned}$$

/65

The next four columns of Table 6 give the values of the functions

$$\begin{aligned}\phi(M) &= \operatorname{arctg} \frac{y-2}{x-2} \\ \bar{\psi}_1(M) &= \sum_{k=1}^{12} \lambda_k \ln r(M_i, M) \\ \bar{\psi}_2(M) &= \sum_{k=1}^{12} a_k^{(A)} \ln r(M_i, M) \\ \bar{\psi}_3(M) &= \sum_{k=1}^{12} a_k^{(\Phi)} \ln r(M_i, M),\end{aligned}$$

where  $M_i$  is the point with coordinates  $x_i = \cos \alpha_i$ ,  $y_i = \sin \alpha_i$  ( $i = 1, 2, \dots, 12$ ) at 12 interior points  $M^{(k)}$  of the ellipse  $\bar{s}$ , with coordinates

$$\begin{aligned}M^{(1)}(0, -0,9), \quad M^{(2)}(0, -0,8), \quad M^{(3)}(0, -0,7), \quad M^{(4)}(0, -0,6), \\ M^{(5)}(0, -0,5), \quad M^{(6)}(0, -0,4), \quad M^{(7)}(0, -0,3), \quad M^{(8)}(0, -0,2), \\ M^{(9)}(0, -0,1), \quad M^{(10)}(0, 0), \quad M^{(11)}(0, 0,1), \quad M^{(12)}(0, 0,2).\end{aligned}$$

From Table 6, it is clear that the approximate solution to the Dirichlet problem analyzed above according to the method of the analyzed series  $\bar{\psi}_2(M)$  gives more exact values than the approximate solution of this same problem by the method of Fourier series  $\bar{\psi}_3(M)$ . Let us mention that  $\phi(M)$  is an exact

solution to this problem and  $\bar{\psi}_1(M)$  is its approximate solution obtained with the aid of the best approximation in the sense of the metric  $L_2$  of the boundary function  $\psi(s)$ .

As noted above, obtaining coefficients (7.4) corresponds to one iteration according to Seidel's method for System (7.2). In this case it is remarkable that we are not required to compute the coefficients of this system -- the scalar products  $(\varphi_i, \varphi_j)$ . This fact, in certain instances, may strongly decrease the number of required quadratures in finding the coefficients of the best approximation, i.e., in solving System (7.21). In fact, let us be required to find the first  $n$  coefficients  $\lambda_i$  in the best approximation of the function  $\psi$  with respect to the system  $\{\varphi_i\}$ . Let us take the new system  $\{\psi_i\}$  ( $i=1, 2, \dots, N; N > n$ ), where  $\psi_k = \varphi_j, k=j \pmod{n}$  and find the coefficients of the proposed series according to this system. It is easy to see that this will correspond to Seidel's iteration process for System (7.21). Here the number of iterations will be equal to  $N/n$ , when  $N = mn$ , where  $m$  is a whole /66 number. A noninteger  $N/n$  means that Seidel's last iteration has not been carried to the end. Since computation of the coefficients of System (7.1) requires  $n^2 + n/2$  quadratures, and computation of  $N$  coefficients (7.4) requires  $N$  quadratures, we may find that use of the method of the proposed series is more convenient than computation of the coefficients of System (7.21) and its further solution [properly, solution to System (7.21) by one of the modifications of the Gauss method still requires on the order of  $n^3$  arithmetic operations]. The last column of Table 3 gives the values of the coefficients  $\bar{a}_k$  obtained when  $N = 36$ , i.e., three iterations were made for System (7.21) by the method of the examined series. The coefficients  $\bar{a}_k$  were computed from the formula  $\bar{a}_k = a_k + a_{12+k} + a_{24+k}$ . The coefficients  $\lambda_k$  and  $\bar{a}_k$  coincide with the five decimal digits. Further increase in the number of iterations made no sense because the scalar products were computed with six reliable decimal digits. Computation of the coefficients  $\bar{a}_k$  required 36 quadratures, whereas computation of the coefficients in System (7.21) would require 78 quadratures.

Let us denote the spectral norm of the operator  $(I - B_1)^{-1}B_2$  by  $\bar{b}$ . Then the number of iterations by the Seidel method for System (7.21) required for decreasing the errors of zero<sup>th</sup> approximation by  $10^k$  times [or what amounts to the same thing, the number of single iterations for System (7.19)] will be equal to  $\frac{k}{\log_{10} \bar{b}}$  and if this number is less than  $n$ , then the iteration process with the aid of the method of the proposed series requires a smaller number of quadratures than computation of the coefficients in System (7.21) and its further solution.

TABLE 4

/67

$i$	$j$	$A_{i,j}$	$i$	$j$	$A_{i,j}$	$i$	$j$	$A_{i,j}$
1	1	1,0407	5	4	- 4,0057	7	6	9,5446
2	1	0,3462	5	5	4,6118	7	7	10,8815
2	2	1,0968	6	1	- 0,9654	8	1	0,8797
3	1	- 0,5365	6	2	1,1276	8	2	1,2645
3	2	- 0,5365	6	3	- 4,3658	8	3	- 11,0209
3	3	0,6252	6	4	1,9865	8	4	- 20,4241
4	1	- 0,3853	6	5	- 1,1420	8	5	13,6851
4	2	- 0,3853	6	6	4,7511	8	6	9,8651
4	3	- 0,2108	7	1	0,9101	8	7	9,6222
4	4	0,6598	7	2	- 0,0561	8	8	14,5257
5	1	0,9226	7	3	- 17,5184	9	1	3,8971
5	2	- 0,7153	7	4	3,3486	9	2	4,5145
5	3	0,9654	7	5	1,0676	9	3	- 135,2959
9	4	- 1,1330	10	8	- 64,2106	11	11	40,7688
9	5	5,7030	10	9	- 58,6615	12	1	5,9396
9	6	33,4046	10	10	177,6990	12	2	2,8762
9	7	- 57,8352	11	1	0,9583	12	3	- 246,1864
9	8	8,3734	11	2	5,2864	12	4	- 95,7584
9	9	167,6996	11	3	- 14,7569	12	5	30,2315
10	1	3,4206	11	4	- 227,4796	12	6	50,5909
10	2	2,5505	11	5	43,0299	12	7	- 225,8198
10	3	46,1256	11	6	14,2266	12	8	- 65,1761
10	4	- 142,9701	11	7	10,4574	12	9	398,9135
10	5	33,0538	11	8	- 216,6044	12	10	139,3438
10	6	- 4,5184	11	9	7,1679	12	11	14,3462
10	7	29,1042	11	10	374,1380	12	12	43,1996

TABLE 5

$i$	$j$	$(\psi_i, \psi_j)$	$i$	$j$	$(\psi_i, \psi_j)$	$i$	$j$	$(\psi_i, \psi_j)$
1	1	0,8607	7	6	0,0243	10	8	0,0338
2	1	0,0905	7	7	0,8804	10	9	— 0,0126
2	2	0,9209	8	1	0,0702	10	10	0,7923
3	1	0,0260	8	2	0,0878	11	1	0,0036
3	2	0,0360	8	3	— 0,0031	11	2	— 0,0631
3	3	0,9422	8	4	— 0,0450	11	3	— 0,0147
4	1	0,0186	8	5	0,0119	11	4	0,0201
4	2	0,0259	8	6	— 0,0074	11	5	— 0,0313
4	3	— 0,0463	8	7	— 0,0041	11	6	0,0150
4	4	0,9735	8	8	0,8732	11	7	0,0315
5	1	— 0,0129	9	1	— 0,0536	11	8	— 0,0186
5	2	0,0227	9	2	— 0,0016	11	9	0,0418
5	3	0,0085	9	3	0,0003	11	10	0,6536
5	4	— 0,0166	9	4	0,0073	11	11	0,7044
5	5	0,8166	9	5	— 0,0077	12	1	— 0,0306
6	1	0,0295	9	6	— 0,0666	12	2	0,0014
6	2	— 0,0108	9	7	0,0373	12	3	0,1002
6	3	— 0,0155	9	8	0,0113	12	4	— 0,0146
6	4	0,0178	9	9	0,7831	12	5	— 0,0036
6	5	0,0109	10	1	0,0020	12	6	— 0,0307
6	6	0,8112	10	2	— 0,0054	12	7	0,0054
7	1	0,0196	10	3	0,0074	12	8	0,0054
7	2	0,0483	10	4	— 0,0047	12	9	0,0564
7	3	— 0,0555	10	5	— 0,0678	12	10	0,0047
7	4	0,0551	10	6	0,0323	12	11	— 0,0169
7	5	— 0,0163	10	7	— 0,0304	12	12	0,6917

TABLE 6

$k$	$a_k^{(\Phi)}$	$a_k^{(A)}$	$\lambda_k$	$\psi(M^{(k)})$	$\bar{\psi}_1(M^{(k)})$	$\bar{\psi}_2(M^{(k)})$	$\bar{\psi}_3(M^{(k)})$	$\bar{a}_k$
1	0,558500	0,423157	0,632909	0,603749	0,518301	0,543001	0,598301	0,632311
2	0,532666	0,681348	0,792195	0,620249	0,562604	0,584391	0,613430	0,792186
3	- 0,855595	- 0,134128	0,485213	0,637549	0,597411	0,601432	0,630724	0,485224
4	2,638250	3,828113	3,946620	0,655696	0,604510	0,621368	0,648858	3,946624
5	1,971488	2,004124	1,801310	0,674741	0,624014	0,641498	0,667883	1,801316
6	2,131344	1,013516	1,163211	0,694738	0,655201	0,664653	0,687857	1,163217
7	4,621686	4,511925	4,613214	0,715944	0,710185	0,709261	0,708842	4,613221
8	0,494495	0,915716	1,121345	0,737815	0,711613	0,713405	0,730897	1,121351
9	- 3,767565	- 3,811601	- 4,267911	0,761013	0,724653	0,734501	0,754085	- 4,267930
10	2,022020	2,561700	2,881903	0,785398	0,734623	0,751908	0,778466	2,881911
11	0,383068	0,981601	1,146823	0,811034	0,754218	0,791423	0,804104	1,146831
12	0,425553	- 0,410116	- 0,495131	0,837981	0,800146	0,814609	0,831662	- 0,495140

## §8. Nonorthogonal Series in Variational Methods

The method proposed in the previous section for solving System (7.17) may be used in variational methods. In essence, expansion of the function  $\varphi(x)$  in a series of functions of the system  $\{\varphi_i(x)\}$  may be regarded as the Ritz method for solving the functional equation  $E\varphi(x) = \varphi(x)$ , which is the identity operator with the coordinate functions  $\{\varphi_i(x)\}$ .

/69

The variational methods of determining the coefficients  $a_j$  of the expansion for solving the functional equations for a certain system  $\{\varphi_j\}$  lead [44] to the following infinite system of equations:

$$\sum_{j=1}^{\infty} (A_1 \varphi_j, A_2 \varphi_k) a_j = (A_3 \varphi_k, A_4 \varphi) \quad (k=1, 2, \dots), \quad (8.1)$$

where  $A_i$  ( $i = 1, 2, 3, 4$ ) are positive or positive definite [44] operators, and  $\varphi$  is a certain known function. If the sequences  $\{A_1 \varphi_j\}$  and  $\{A_2 \varphi_j\}$  are biorthonormalized or, when  $A_1 \equiv A_2$ , the sequence  $\{A_1 \varphi_j\}$  is orthonormalized, then for the coefficients  $a_j$  we obtain

$$a_j = (A_3 \varphi, A_4 \varphi_j). \quad (8.2)$$

In certain instances [22], however, the system  $\{A_1 \varphi_j\}$  is not strongly minimal and its preliminary orthonormalization involves great difficulties [23]. Therefore, the question arises of approximation solution of System (8.1). By approximate solution of System (8.1) we shall mean the vector  $\bar{a}^{(N)} = (\bar{a}_1^{(N)}, \dots, \bar{a}_N^{(N)})$ , which satisfies the system

$$A^{(N)} \bar{a}^{(N)} = B^{(N)} + \varepsilon \quad (8.3)$$

or

$$\sum_{j=1}^N (A_1 \varphi_j, A_2 \varphi_k) \bar{a}_j^{(N)} = (A_3 \varphi, A_4 \varphi_k) + \varepsilon_k \quad (k=1, 2, \dots, N),$$

where  $\varepsilon(\varepsilon_1, \dots, \varepsilon_N)$  is the discrepancy vector. For smallness of the vector  $(a^{(N)} - \bar{a}^{(N)})$ , where  $a^{(N)}$  is the solution to system (8.3) when  $\varepsilon \equiv 0$ ,  $a_i^{(N)}$  are coefficients of the best, in a certain sense, expansion of the solution to the functional equation for the system  $\{\varphi_i\}$ , it is necessary that the expression  $\|(A^{(N)})^{-1}\varepsilon\|$  be small. However, in the case of unreliable systems (which are not a Barry basis [39]) the norm of the vector  $(a^{(N)} - \bar{a}^{(N)})$  may be as large as desired but nevertheless the difference between the expanded function and its approximation  $\sum_{i=1}^N a_i^{(N)} \varphi_i$  may be in a certain metric less than the small number  $\varepsilon > 0$  as much as is desired

$$\left\| \psi - \sum_{i=1}^N a_i^{(N)} \varphi_i \right\| < \varepsilon. \quad (8.4)$$

In the latter case the respective matrix is poorly defined and in its vicinity we find a degenerate matrix. As shown in [45] without regularization here we can obtain greatly differing solutions which differ from the normal solution [45].

Therefore, we shall term the vector  $\bar{a}^{(N)}$ , which satisfies inequality (8.4)/70 the  $\varepsilon$ -approximate solution to System (8.1).

In the present section we can indicate one new method for obtaining the approximate solution to System (8.1). The essence of this new method involves the fact that in the case of certain systems  $\{\varphi_i\}$  (the respective conditions for the systems will be formulated below) for obtaining the approximate solution to (8.1) with a large  $N$  it is sufficient to make only one Seidel iteration, taking as the initial approximation the null vector.

In the Hilbert space  $H$ , in which the scalar product  $[u, v]$  is determined in the following manner:

$$[u, v] = (A_1 u, A_2 v) = (A_1 v, A_2 u),$$

let us seek the best, in the sense of the metric  $H$ , approximate functions  $\psi$  with the generalized polynomial  $\sum_{j=1}^n a_j \varphi_j$ . From the orthogonality of the difference  $(\psi - \sum_{j=1}^n a_j \varphi_j)$  to any function  $\varphi_i$  we obtain for determination of the coefficients  $a_j$  the following system:

$$\left[ \psi - \sum_{j=1}^N a_j \varphi_j, \varphi_k \right] = 0 \quad (k=1, 2, \dots, N),$$

or

$$\sum_{j=1}^N a_j (A_1 \varphi_j, A_2 \varphi_k) = (A_1 \psi, A_2 \varphi_k).$$

If the operator  $A_1$  is the product  $A_1 = A_5 A_4 A_6$ , where  $A_5$  satisfies the condition

$$(A_5 A_4 \varphi, A_2 \varphi_i) = (A_4 \varphi, A_6 A_2 \varphi_i),$$

and the functional equation

$$A_6 \psi = \varphi,$$

is given, then by denoting  $A_5 A_2 = A_3$ , for determining the coefficients  $a_j$  we find the system

$$\sum_{j=1}^N a_j (A_1 \varphi_j, A_2 \varphi_k) = (A_3 \varphi_k, A_4 \varphi).$$

We shall determine the approximate values  $\bar{a}_j$  of the coefficients  $a_j$  using Seidel's iteration process after taking the vector  $a^0 = (0, \dots, 0)$  as the initial approximation

$$\begin{aligned} \bar{a} &= \frac{(A_3 \varphi_1, A_4 \varphi)}{(A_1 \varphi_1, A_2 \varphi_1)}, \\ \bar{a}_2 &= \frac{(A_3 \varphi_2, A_4 \varphi) - \bar{a}_1 (A_1 \varphi_1, A_2 \varphi_2)}{(A_1 \varphi_2, A_2 \varphi_2)}, \\ \bar{a}_h &= \frac{(A_3 \varphi_h, A_4 \varphi) - \sum_{j=1}^{h-1} \bar{a}_j (A_1 \varphi_j, A_2 \varphi_h)}{(A_1 \varphi_h, A_2 \varphi_h)}. \end{aligned}$$

/71

Let us note that if the systems  $\{A_1 \varphi_j\}$ ,  $\{A_2 \varphi_j\}$  are biorthonormalized, then the coefficients  $\bar{a}_j$  coincide with the coefficients (8.2) of the function  $\psi$ .

Since

$$(A_1 \psi, A_2 \varphi_k) = (A_2 \varphi_k, A_1 \psi),$$

then for  $\bar{a}_k$  we obtain

$$\bar{a}_k = \frac{\left( A_1 \left( \psi - \sum_{j=1}^{k-1} \bar{a}_j \varphi_j \right), A_2 \varphi_k \right)}{(A_1 \varphi_k, A_2 \varphi_k)}.$$

Thus,  $\bar{a}_k$  is a coefficient of the best approximation (in the sense of the examined Hilbert space H), of the difference

$$\begin{aligned} \left( \psi - \sum_{j=1}^{k-1} \bar{a}_j \varphi_j \right) &= \varphi^{(k-1)} \text{ as a function } C_k \varphi_k \text{ [10]} \\ &\text{of} \end{aligned} \quad (8.5)$$

$$\min_{c_k} \left\| \psi - \sum_{j=1}^{k-1} \bar{a}_j \varphi_j - c_k \varphi_k \right\| = \left\| \psi - \sum_{j=1}^k \bar{a}_j \varphi_j \right\| = \frac{G(\varphi^{(k-1)}, \varphi_k)}{G(\varphi_k)},$$

where  $G(u_1, \dots, u_n)$  is the Gram determinant of the function  $u_1, \dots, u_n$ . Since

$$\frac{G(\varphi^{(k-1)}, \varphi_k)}{G(\varphi_k)} = (\|\varphi^{(k-1)}\|)^2 = \frac{(A_1 \varphi^{(k-1)}, A_2 \varphi_k)^2}{\|\varphi_k\|^2},$$

then from Expression (8.5) we obtain

$$(\|\varphi^{(k)}\|)^2 = (\|\varphi^{(k-1)}\|)^2 - \frac{(A_1 \varphi^{(k-1)}, A_2 \varphi_k)^2}{\|\varphi_k\|^2}. \quad (8.6)$$

Thus, the sequence of positive numbers  $\{\|\varphi^{(k)}\|\}$  is monotonically decreasing and consequently has a limit. If we assume that the norm of the

functions  $\varphi_i$  are bounded in the set both from above and from below, from (8.6) we find that for any  $\varepsilon > 0$  we find an  $N_0$  such that

$$\sum_{k=N_0}^{\infty} (A_1 \varphi^{(k-1)}, A_2 \varphi_k)^2 < \varepsilon. \quad (8.7)$$

But then when  $S > N_0$ ,

$$\begin{aligned} \varepsilon &> |(\|\varphi^{(s)}\|)^2 - (\|\varphi^{(s+1)}\|)^2| = |(A_1(\psi - \sum_{j=1}^s a_j \varphi_j), A_2(\psi - \sum_{j=1}^s a_j \varphi_j)) - \\ &\quad - (A_1(\psi - \sum_{j=1}^{s+1} a_j \varphi_j), A_2(\psi - \sum_{j=1}^{s+1} a_j \varphi_j))| = |(A_1(\psi - \sum_{j=1}^s a_j \varphi_j), \\ &\quad A_2(\psi - \sum_{j=1}^s a_j \varphi_j)) - a_{s+1}(A_1 \varphi_{s+1}, A_2(\psi - \sum_{j=1}^s a_j \varphi_j)) - a_{s+1}^2(A_1 \varphi_{s+1}, A_2 \varphi_{s+1}) - \\ &\quad - (A_1(\psi - \sum_{j=1}^s a_j \varphi_j), A_2(\psi - \sum_{j=1}^s a_j \varphi_j))| = |-3a_{s+1}^2(A_1 \varphi_{s+1}, A_2 \varphi_{s+1})|, \end{aligned}$$

and on the strength of the boundedness of the norm of the functions  $\varphi_j$  from 72 below we have

$$|a_{s+1}| < \varepsilon.$$

From the latter inequality we find that for any  $\varepsilon > 0$  and any  $N$  we find an  $N_0$  such that

$$\varphi^{(s)} = \varphi^{(r)} + \gamma_r^{(s)} \quad (N \leq s, r \leq N_0 + N), \quad (8.8)$$

where

$$\|\gamma_r^{(s)}\| < \varepsilon. \quad (8.9)$$

Substituting (8.8) into (8.7) we obtain

$$\begin{aligned} \varepsilon &> \sum_{k=N_0}^{\infty} (A_1 \varphi^{(k-1)}, A_2 \varphi_k)^2 > \sum_{k=N_0}^{N_0+N} (A_1 \varphi^{(k)}, A_2 \varphi_{k-1})^2 = \sum_{k=N_0}^{N_0+N} (A_1 \varphi^{(r)}, A_2 \varphi_{k-1})^2 + \\ &\quad + \sum_{k=N_0}^{N_0+N} (A_1 \gamma_r^{(k)}, A_2 \varphi_{k-1})^2 + 2 \sum_{k=N_0}^{N_0+N} (A_1 \varphi^{(r)}, A_2 \varphi_{k-1}) (A_1 \gamma_r^{(k)}, A_2 \varphi_{k-1}). \end{aligned}$$

But taking (8.9) into account, we find that for any  $\varepsilon > 0$  and a whole  $N$  we find an  $N_0$  such that

$$\sum_{k=N_0}^{N_0+N} (A_1 \varphi^{(r)}, A_2 \varphi_{k-1})^2 < \varepsilon \quad (N_0 < r < N_0 + N). \quad (8.10)$$

We shall assume that the system  $\{\varphi_i\}$  satisfies the condition: for any  $\varepsilon > 0$  of a whole  $N$  and  $\psi \in H$  we find coefficients  $b_k$  ( $k = N_0, \dots, N_0 + N$ ) such that at least for one value of  $r$

$$\|\varphi^{(r)} - \sum_{k=N_0}^{N_0+N} b_k \varphi_k\| < \varepsilon \quad (N_0 < r < N_0 + N) \quad (8.11)$$

and

$$\sum_{k=N_0}^{N_0+N} b_k^2 < M, \quad (8.12)$$

where  $M$  is a constant, independent of  $N_0$  and  $N$ .

On the strength of (8.10), (8.11), (8.12) and the Schwarz-Buniakowski inequality, we obtain

$$\left| \sum_{k=N_0}^{N_0+N} b_{k-1} (A_1 \varphi^{(r)}, A_2 \varphi_{k-1}) \right| < \varepsilon_1.$$

Taking into account the last two inequalities and the fact that  $\|\varphi^{(r)}\| < \|\psi\|$  /73 we obtain

$$\begin{aligned} \|\psi - \sum_{j=1}^r \bar{a}_j \varphi_j\|^2 &= \|\varphi^{(r)}\|^2 = (A_1 \varphi^{(r)}, A_2 \sum_{k=N_0}^{N_0+N} b_k \varphi_k) - \\ &- \left( A_1 \varphi^{(r)}, A_2 \left( \sum_{k=N_0}^{N_0+N} b_k \varphi_k - \varphi^{(r)} \right) \right) < \varepsilon_1 + \|\varphi^{(r)}\| \|\varphi^{(r)} - \sum_{k=N_0}^{N_0+N} b_k \varphi_k\| \\ &- \sum_{k=N_0}^{N_0+N} b_k \varphi_k\| < \varepsilon_1 + \|\psi\| \varepsilon < \varepsilon_2, \end{aligned}$$

where  $\varepsilon_2$  is a number as small as desired.

Thus, we have proved the following theorem.

If the norms of the function  $\varphi_j$  in Hilbert space  $H$  are bounded both from above and below the system  $\{\varphi_j\}$  and the function  $\psi$  satisfy Conditions (8.11) and (8.12), then for the  $\varepsilon$ -approximate solution to System (8.1) it is sufficient to carry out a single Seidel iteration for the system

$$\sum_{j=1}^{N(\varepsilon)} (A_1 \varphi_j, A_2 \varphi_k) a_j = (A_3 \varphi_k, A_4 \varphi) \quad (k=1, 2, \dots, N(\varepsilon)),$$

taking as the initial approximation the null vector  $a^0(0, 0, \dots, 0)$ .

#### §9. Approximate Solution to One Mixed Boundary Value Problem in the Theory of Harmonic Functions

Let us analyze the boundary value problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } G, \\ u|_{\Gamma_i} &= \omega(s) + c_i, \\ \int_{\Gamma_i} \frac{\partial u}{\partial n} ds &= 0 \quad (i=1, 2, \dots, m), \end{aligned} \tag{9.1}$$

where  $G$  is a multiply connected two-dimensional (three-dimensional) region (Figure 1), bounded by the contour (surface)  $\Gamma = \sum_{i=1}^m \Gamma_i$ , and  $G_i$  are hitherto unspecified constants. If for the holomorphic function  $g(z) = u + iv$ , the Cauchy-Riemann expression is valid

$$\frac{\partial v}{\partial s} = \frac{\partial u}{\partial n} \tag{9.2}$$

also on the boundary of the region  $G$ , then Problem (9.1) in the two-dimensional case coincides with the so-called modified Dirichlet problem [46], i.e., to find the function  $u(x, y)$  which is harmonic in  $G$  and continuous in  $G + l$  under the following conditions: (1)  $u(x, y)$  is the real part of the function  $g(z)$  which is holomorphic in  $G$ ; (2) it satisfies the boundary condition

$u = \omega(s) + c_i$ , where  $\omega(s)$  is a continuous function, and  $c_i$  are real constants, hitherto not given<sup>(11)</sup>. /74

This problem is a rather wide-spread one. It is encountered in the approximate solution to boundary value problems in the theory of analytical functions [42], in the conformal mapping of multiply connected regions [48], in computing fields of charged filaments located near conducting cylinders [49], etc.

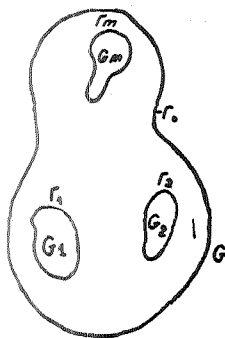


Figure 1

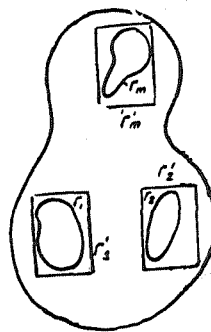


Figure 2

In [46], it was proven that the modified Dirichlet problem with one of the arbitrary constants  $c_i$  specified has a unique solution which can be represented in the form of a double layer<sup>(12)</sup> potential. The integral equation for the density in this case can be assigned such a form that the

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(11) In order that the Problem (9.1) coincide with the modified Dirichlet problem [46], instead of Condition (9.2) it is sufficient to require satisfaction of the condition  $\int \frac{\partial v}{\partial s} ds = \int \frac{\partial u}{\partial n} ds$ .

(12) This solution satisfies Condition (9.2) in the closed region  $\bar{G} = G + S$ . Hence, as a result of the uniqueness of the adjoint function  $v(x, y)$  we have  $\int_{\Gamma_i} \frac{\partial u}{\partial n} ds = 0$ . On the strength of the uniqueness theorem for Problem (9.1), this means that in the two-dimensional case Problem (9.1) is equivalent to the modified Dirichlet problem.

respective homogenous equation will not have a nonzero solution. However, practical solution to these integral equations, in spite of their Fredholm character (in the case of regions bounded by Lyapunov curves) is made more difficult, since the kernel has a variable indeterminacy of the type 0:0 and this limits the accuracy of computing coefficients of the linear system by which the integral equation is replaced in the approximate solution.

In [47] for the approximate solution to Problem (9.1) (when  $c_m = 0$ ) it is proposed to use the method of finite differences. The integral conditions at the boundary are replaced by the finite-difference approximation, and instead of the contours  $\Gamma_i$ , we consider the rectangular contours  $\Gamma'_i$  (Figure 2) and the conditions

$$\int_{\Gamma_i} \frac{\partial u}{\partial n} ds = 0.$$

/75

Hence, from the harmonicity of  $u$  there follow the integral conditions (9.1). Such a replacement of the contours represents definite convenience from the viewpoint of programming the solution to the respective finite-difference system. However, we must mention that the matrix of this system may be singular in this case (in [47] the question was not investigated as to the solvability and uniqueness of the solution to the obtained finite-difference system). Therefore, these integral conditions are more feasible to analyze namely in the form of (9.1), i.e., on  $\Gamma_i$ . For this from all the interior points of the region, the distances from which to the boundary  $\Gamma_i$  are less than  $\nu h$ , where  $h$  is the grid-point spacing and  $\nu$  is a certain constant (for example,  $\nu < \sqrt{2}$ ), we must drop the normal on  $\Gamma_i$  and the points of intersection of this normal with  $\Gamma_i$  must be analyzed as points of the quadratic formula for the integrals (1)<sup>(13)</sup>. It is easy to prove that

---

(13) The other terms in these formulas will be one order of magnitude rougher than in the formulas from Reference [47]. However, it does not follow from this that the error in solving the proposed system will be greater than the error in solving the system used in [47].

the matrix thus obtained of the finite-difference system on the strength of its indivisibility (with a sufficiently small  $h$ ) will satisfy the conditions of the familiar theorem of Olga Tausski [50] on the nonsingularity of the matrices (it is assumed that at the interior and boundary points, regular finite-difference approximations are used), namely the diagonal term in each row in absolute value will be no less than the sum of the absolute values of all the nondiagonal terms and in some rows (for the points near the contour  $\Gamma_m$ ) the diagonal term will be, in absolute value, strictly greater than the sum of the absolute values of all the nondiagonal terms.

For practical solution of this system it is necessary to indicate (if possible) the converging iteration process, because the elliptical finite-difference systems as a rule are solved by iteration methods. Reference [49] gives an algorithm for solving Problem (9.1) when  $m = 2$  and  $c_2 = 0$  (doubly connected region), which assumes knowledge of the function  $\Lambda(z)$ , which is infinitely sheeted, and conformal mapping of the given doubly connected region onto a strip. This algorithm is quite complex and is feasible to be used only in those cases when  $\Omega(z)$  is written explicitly (for example, when the doubly connected region is a ring [49]).

We can show that the solution to Problem (9.1) can be constructed with comparative simplicity by the methods analyzed above.

Let  $u_i$  ( $i = 1, \dots, m$ ) be the solutions to the following boundary value problems.:

$$\begin{aligned} \Delta u_i &= 0 \text{ в } G, \quad u_i|_{\Gamma_j} = \delta_{i,j} \quad (i=1, \dots, m), \\ \Delta u_{m+1} &= 0 \text{ в } G, \quad u_{m+1}|_{\Gamma} = \omega(s), \end{aligned} \quad (9.3)$$

where  $\delta_{i,j}$  is the Kronecker symbol. Solution to Problem (1) will be sought in the following form:

$$u = \sum_{i=1}^{m+1} c_i u_i,$$

where  $c_{m+1} = 1$ . Then to determine the constants  $c_i$  ( $i = 1, \dots, m$ ) from the integral Conditions (1) we obtain a system of equations

$$\sum_{i=1}^m c_i r_{i,j} = \int_{\Gamma_j} \frac{\partial u_{m+1}}{\partial n} ds \quad (j=1, 2, \dots, m), \quad (9.4)$$

where

$$\begin{vmatrix} -\int_{\Gamma_1} \frac{\partial u_1}{\partial n} ds, & -\int_{\Gamma_1} \frac{\partial u_2}{\partial n} ds, & \dots, & -\int_{\Gamma_1} \frac{\partial u_m}{\partial n} ds \\ -\int_{\Gamma_2} \frac{\partial u_1}{\partial n} ds, & -\int_{\Gamma_2} \frac{\partial u_2}{\partial n} ds, & \dots, & -\int_{\Gamma_2} \frac{\partial u_m}{\partial n} ds \\ \dots & \dots & \dots & \dots \\ -\int_{\Gamma_m} \frac{\partial u_1}{\partial n} ds, & -\int_{\Gamma_m} \frac{\partial u_2}{\partial n} ds, & \dots, & -\int_{\Gamma_m} \frac{\partial u_m}{\partial n} ds \end{vmatrix} = (r_{i,j})$$

$$(r_{i,j} = -\int_{\Gamma_i} \frac{\partial u_j}{\partial n} ds).$$

From the principle of the maximum it follows that for the inner normal we will have

$$\frac{\partial u_i}{\partial n} \Big|_{\Gamma_i} < 0, \quad \frac{\partial u_i}{\partial n} \Big|_{\Gamma_j} > 0 \quad (i, j=1, 2, \dots, m).$$

We know (see [52], page 20) that with sufficiently smooth contours (surfaces)  $\text{grad } u_i \Big|_{\Gamma_j} \neq 0$  ( $i, j=1, 2, \dots, m$ ). But since  $u_i \Big|_{\Gamma_j} = 0$  ( $i \neq j$ ), then

$$\text{grad } u_i \Big|_{\Gamma_j} = \frac{\partial u_i}{\partial n} \Big|_{\Gamma_j} > 0 \quad (i \neq j).$$

From the latter inequalities and relationships

$$\int_{\Gamma} \frac{\partial u_i}{\partial n} ds = 0 \quad (i=1, 2, \dots, m)$$

we find

$$r_{i,j} = \sum_{\substack{i=1 \\ i \neq j}}^m |r_{i,j}| = - \sum_{\substack{i=1 \\ i \neq j}}^m r_{i,j},$$

which indicates the singularity of the matrix  $(r_{i,j})$ . However, after arbitrarily stipulating one of the constants  $c_i$  and eliminating the corresponding equation from System (9.4), it is easy to see that the respective matrix R will satisfy the familiar Adamar condition relative to nonsingularity of the matrices (the diagonal term in any column is greater in absolute value than the sum of the absolute values of all the other terms [50]). Since for the Adamar type matrix R the conditions  $r_{i,i} > 0$  ( $i=1, 2, \dots, m$ ), are also valid, then the determinant of the matrix is positive (see [50], page 26). /77

These arguments prove the uniqueness of the solution to Problem (9.1) with one of the constants  $c_k$  ( $1 \leq k \leq m$ ) arbitrarily specified. Let us denote the elements of the inverse matrix  $R^{-1}$  by  $(\alpha_{i,j})$ . On the strength of the nonsingularity  $R \sum_{i,j} |\alpha_{i,j}| = \sigma < \infty$ . From Ostrovskiy's results (see [50], page 42) it follows that the matrix of System (9.4) will also be nonsingular for such approximate solutions  $\bar{u}_i$  to Problems (9.3) which satisfy the condition

$$\max_{i,j} \left| \varepsilon_{i,j} \right| = \max_{i,j} \left| \int_{\Gamma_j} \frac{\partial}{\partial n} (u_i - \bar{u}_i) ds \right| = \varepsilon < \frac{1}{\sigma} < \frac{1}{(m-1)^2 \alpha},$$

where  $\alpha = \max_{i,j} |\alpha_{i,j}|$ . Let us also note that the matrix R depends only on the geometry of the given region and does not depend on the boundary conditions. Therefore, for the specific region we can once and forever compute  $\alpha$ .

Thus, to obtain an approximate solution to Problem (9.1) we must specify one of the constants  $c_i$ , to approximately solve Problem (9.3) and use the system obtained from (9.4) by discarding one equation. Here the approximate solutions of Problems (9.3) must satisfy the following conditions: (1) for any  $\varepsilon > 0$  we can construct an approximate solution which satisfies the condition:

$$\max_{i,j} \left| \int_{\Gamma_j} \frac{\partial}{\partial n} (u_i - \bar{u}_i) ds \right| < \varepsilon$$

(obviously for this it is sufficient that the normal derivative  $\frac{\partial \bar{u}_i}{\partial n}$  of the approximate solution approximate  $\frac{\partial \bar{u}}{\partial n}$  in the sense of the metric  $L_2$ ); (2) furthermore, it is desirable that the approximate method for solving Problems (9.3) be such that, after obtaining the solution to one of the Problems in (9.3), the solutions to the other problems will be obtained automatically, without considerable computation. These requirements are satisfied by the method of solving the boundary value problems described above. Let us analyze the following linearly independent system of functions [3], complete in  $L_2$  at  $\Gamma$ :

$$\{\ln r(x_i, y)\}, \quad (9.5)$$

where  $x_i$  are the elements of the denumerable set of points distributed everywhere densely on the auxiliary contour  $s = \sum_{i=0}^m s_i$  (Figure 3). Below we analyze Problem (9.1) for the two-dimensional case although all the arguments obviously are /78 valid also for the case of three dimensions; here System (9.5) must be replaced by the system  $\{[\ln r(x_i, y)]^{-1}\}$ , where  $x_i$  are distributed on a certain surface [6].

Let us consider the system  $\{\varphi^{(i)}(y)\}$ ;

$$\varphi^{(i)}(y) = \sum_{j=1}^i A_{j,i} \ln r(x_j, y),$$

where  $A_{j,i}$  are the coefficients of orthonormalization of System (9.5)

Using for  $u_{m+1}$  the Green formula at the points  $x_i$ , we find

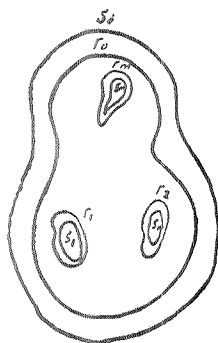


Figure 3

where

$$\begin{aligned} \int_{\Gamma} \ln r(x_i, y) \varphi_{m+1}(y) ds_y &= F_i = \\ &= \int_{\Gamma} u_{m+1}(s) \frac{\partial}{\partial n_y} \ln r(x_i, y) ds_y, \end{aligned} \quad (9.6)$$

After multiplying the first  $i$  in Equation (9.6) by  $A_{j,i}$  ( $j = 1, 2, \dots, i$ ) and combining, we find

$$\int_{\Gamma} \varphi_{m+1}(y) \varphi^{(i)}(y) ds_y = \Phi_i^{(m+1)} = \sum_{j=1}^i A_{j,i} F_j, \quad (9.7)$$

where  $\Phi_i^{(m+1)}$  are the Fourier coefficients of the unknown function  $\varphi_{m+1}(y)$ . Assuming that  $\varphi_{m+1}(y) \in L_2$  and taking into account the completeness of the system  $\{\varphi^{(i)}(y)\}$ , we will have

$$\lim_{N \rightarrow \infty} \left\| \varphi_{m+1}(y) - \sum_{i=1}^N \Phi_i^{(m+1)} \varphi^{(i)}(y) \right\|_{L_2} = 0,$$

Let us note that the approximate values of the right-hand sides of System (9.4) have been found (even before Problem (9.3) was solved for  $u_{m+1}$ ).

Let us analyze  $(m)$  classes  $W_i$  ( $i = 1, 2, \dots, m$ ) of numbers according to absolute value  $(m)$ . We shall assume that the points  $x_i$  are distributed such that if  $k \in W_i$ , then  $x_k \in S_i$ .

Let us use the Green formula for  $u_i$  ( $i = 1, 2, \dots, m$ ) at the points  $x_k$ :

$$\int_{\Gamma} \ln r(x_k, y) \varphi_i(y) ds_y = 2\pi \text{ when } k \in W_i,$$

$$\int_{\Gamma} \ln r(x_k, y) \varphi_i(y) ds_y = 0 \text{ when } k \in W_j \ (j \neq i),$$

where

$$\varphi_i(y) = \frac{\partial u_i}{\partial n} \Big|_{\Gamma}.$$

For the coefficients of the generalized Fourier series of the function  $\varphi_i(y)$  we obtain

$$\Phi_k^{(i)} = 2\pi \sum_{j=1}^r A_{jm+i, k}, \quad (9.8) \quad \underline{/79}$$

where  $r$  is the greatest whole number which satisfies the condition  $rm+i < k$ .

Expressions (9.7) and (9.8) make it possible to obtain the right-hand sides and coefficients of System (9.4), the solutions  $c_i$  of which (for one specified constant), are introduced into the Green formula for solving Problem (9.1):

$$u(x) = \frac{1}{2\pi} \sum_{k=1}^m \int_{\Gamma_k} [\omega(s) + c_k] \frac{\partial}{\partial n} \ln r(x, y) ds_y -$$

$$- \frac{1}{2\pi} \int_{\Gamma} \ln r(x, y) \varphi(y) ds_y, \quad x \in G, \quad (9.9)$$

where

$$\varphi(y) = \frac{\partial u}{\partial n} \Big|_{\Gamma} = \sum_{i=1}^{m+1} c_i \sum_{j=1}^i \Phi_j^{(i)} \varphi^{(i)}, \quad c_{m+1} = 1,$$

give values of the solution to Problem (9.1) at the arbitrary point  $x \in G$ .

In the two-dimensional case, after computing  $u(x,y)$ , it is often required [44], [48] to also compute the imaginary part  $v(x,y)$  of the holomorphic function  $g(z) = u + iv$ . For this we can use the algorithm from Reference [51], which assumes computation of the values  $v$  at the points at the limit by recurrence relationships (difference analog of the Cauchy Riemann relationships) and at the interior points by solving the system of difference equations. However, knowledge of the functions  $u|_{\Gamma}$  and  $\frac{\partial u}{\partial n}|_{\Gamma}$  makes it possible to compute the functions

$$v(y)|_{\Gamma} = v_0 + \int_0^y \frac{\partial v}{\partial s} ds = v_0 + \int_0^y \frac{\partial u}{\partial n} ds,$$

$$\frac{\partial v}{\partial n}|_{\Gamma} = \frac{\partial u}{\partial s} = w'_s(s),$$

and then for computation of  $v(x)$   $x \in G$ , the Green Formula (9.9) is used.

#### §10. Approximate Solution to the Reimann-Hilbert Problem

The Riemann-Hilbert problem for a multiply connected region is the following. In the region  $G_i$  (Figure 1), bounded by the contour  $\Gamma = \Gamma_0 + \Gamma_1 + \dots + \Gamma_m$ , we seek the holomorphic function  $F(s) = u + iv$ , which satisfies at the contour  $\Gamma$  of the region  $G_i$  the condition

$$\alpha(s)u + \beta(s)v = \gamma(s).$$

Assuming that  $\alpha(s)$ ,  $\beta(s)$ ,  $\gamma(s)$  are continuous Hölder functions with an arc length  $S$  on the boundary  $\Gamma$  and  $\alpha^2 + \beta^2 = 1$ , and  $F(z)$  is a continuous holomorphic Hölder function in the closed region  $G(F(z) - H_0$  [47]), the solution to the formulated problem when  $n+1 > m$ , where  $n$  is the index of the analyzed problem [53] (the case  $n+1 < m$  can [47] be reduced to the problem when  $n = m$ ), is reduced [47] to solving the following boundary value problems (we

/80

know [53], that when  $n+1 > m$  the analyzed problem is always solvable, and the respective homogeneous problem has  $(2n+1-m)$  linearly independent solutions.

I. We seek  $H_0$  - the holomorphic function  $g(z)$  assuming the following conditions at the boundary:

$$\begin{aligned} \operatorname{Re} g(z) |_{\Gamma_0} &= \omega(s), \\ \operatorname{Re} g(z) |_{\Gamma_k} &= \omega(s) + C_k \quad (k=1, 2, \dots, m), \end{aligned}$$

where  $\omega(s)$  is a known function, and  $C_k$  are arbitrary constants. Here we shall assume for generality that  $p$  ( $p \leq m$ ) of them (numeration of the contours is done in such a manner that they are the first  $p$ ) are equal to zero. This problem, as noted above, is also encountered [48] in conformal mapping of multiply connected regions and in computing [49] the field of charged filaments distributed near conducting cylinders.

II. We seek the function which is harmonic in  $G_1$  assuming the following boundary values:

$$\begin{aligned} u(s) |_{\Gamma_k} &= \psi(s) \quad (k=0, 1, \dots, p), \\ \left( \frac{\partial u}{\partial n} + \gamma_k \frac{\partial u}{\partial s} \right) |_{\Gamma_k} &= \varphi(s) \quad (k=p+1, \dots, m), \end{aligned}$$

where  $\psi(s)$  and  $\varphi(s) = \frac{\partial \psi}{\partial s}$  are known functions, and  $\gamma_k$  are nonzero constants.

III. We seek the function  $v$  which is harmonic in  $G_1$  with the following boundary values:

$$\begin{aligned} \frac{\partial u}{\partial n} |_{\Gamma_k} &= -\varphi(s) \quad (k=0, 1, \dots, p), \\ \left( \gamma_k \frac{\partial u}{\partial n} - \frac{\partial u}{\partial s} \right) |_{\Gamma_k} &= -\varphi(s) \quad (k=p+1, \dots, m). \end{aligned}$$

The general solution to the Riemann-Hilbert problem with  $(2n + 1 - m)$  arbitrary coefficients (their number may be equal to zero for the region of odd connectedness) is constructed [47] from a solution to these boundary value problems. To obtain some partial solution, it is necessary to have additional information relative to the unknown function (see [53], Chapter 4, §6).

Let us analyze Problem I. It was analyzed in the previous section and therefore we shall not touch upon the details of the cited algorithm. We know [53] that for uniqueness of  $H_0$  - the holomorphic function,  $g(z) = u + iv$ , where

$$\begin{aligned} \Delta u &= 0 \text{ in region } G_i, \\ u \Big|_{\Gamma_0} &= \omega(s), \\ u \Big|_{\Gamma_k} &= \omega(s) + G_k \quad (k=1, 2, \dots, m), \end{aligned} \tag{10.1} \quad \underline{/81}$$

it is necessary and sufficient that the constants  $G_k$  be selected in such a way that

$$\int_{\Gamma_k} \frac{\partial v}{\partial s} ds = 0 \quad (k=1, 2, \dots, m)$$

or taking into account the Cauchy-Riemann relationships we find for determination of  $G_k$  the following conditions:

$$\int_{\Gamma_k} \frac{\partial u}{\partial n} ds = 0 \quad (k=0, 1, \dots, m). \tag{10.2}$$

Equation (10.2) when  $k = 0$  is a simple corollary of the harmonicity of  $u$  and Equation (10.2) when  $k = 1, 2, \dots, m$ .

In the preceding section for solving problem (10.1) - (10.2) we solved  $m$  boundary value Problems (9.3). In the present section we give formulas for solving Problem (10.1) - (10.2), which do not require preliminary solution to all problems in (9.3).

For solution to Problem (10.1) - (10.2) the methods developed above are well suited because in the course of solving the problems by these methods the function  $\frac{\partial u}{\partial n}$  is computed automatically.

Using the Green formula for solving Problem (10.1), we find (below we will use the notations in Reference [3])

$$u(x) = \frac{1}{2\pi} \sum_{k=0}^m \int_{\Gamma_k} [\omega(s) + C_k] \frac{\partial}{\partial n_y} \ln r(x, y) ds_y - \frac{1}{2\pi} \int_{\Gamma} \ln r(x, y) \varphi(y) ds_y, \quad x \in G_i, \quad (10.3)$$

$$0 = \frac{1}{2\pi} \sum_{k=0}^m \int_{\Gamma_k} [\omega(s) + C_k] \frac{\partial}{\partial n_y} \ln r(x, y) ds_y - \frac{1}{2\pi} \int_{\Gamma} \ln r(x, y) \varphi(y) ds_y, \quad x \in G_e, \quad (10.4)$$

where

$$G_e = \sum_{k=0}^m G_k \quad (\text{Fig. 3}) \quad \varphi(y) = \frac{\partial u}{\partial n} \Big|_{\Gamma}, \quad c_0 = 0. \quad (10.5)$$

Taking into account the notation

$$\int_{\Gamma} \omega(s) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y = F(x), \quad \underline{182}$$

and the equation

$$\int_{\Gamma_x} \frac{\partial}{\partial n_y} \ln r(x, y) ds_y = \begin{cases} 2\pi & \text{for } x \in G_k \\ 0 & \text{for } x \in G_e - G_k, \end{cases}$$

Equation (10.4) takes the form

$$\int_{\Gamma} \ln r(x, y) \varphi(y) ds_y = F(x) + 2\pi G, \quad x \in G_k. \quad (10.6)$$

Let us analyze the system  $\{\varphi_i(y)\}$  [3], obtained by orthonormalization of the linearly independent and complete<sup>(14)</sup> system  $\{\ln r(x_i, y)\}$ , where  $x_i$  are elements of the denumerable set, distributed everywhere dense on the auxiliary [3] boundary  $s = s_0 + s_1 + s_2 + \dots + s_m$  (Figure 3)

$$\varphi_i(y) = \sum_{j=1}^i A_{j,i} \ln r(x_j, y),$$

where  $A_{j,i}$  are the coefficients of orthonormalization.

Let us write (10.6) for the points  $x_i$

$$\int_{\Gamma} \ln r(x_i, y) \varphi(y) ds_y = F_i + 2\pi C^{(i)}, \quad x_i \in G, \quad (10.7)$$

where  $F_i = F(x_i)$ ,  $C^{(i)} = C_k$ , if  $x_i \in S_k$ . Multiplying the first  $i$  in the equation by  $A_{j,i}$  ( $j = 1, 2, \dots, i$ ) and combining them

$$\int_{\Gamma} \varphi(y) \varphi_i(y) ds_y = \Phi_i = \sum_{j=1}^i A_{j,i} F_j + 2\pi \sum_{j=1}^i C^{(j)} A_{j,i} = \bar{\Phi}_i + \pi \sum_{j=1}^i C^{(j)} A_{j,i}, \quad (10.8)$$

where  $\Phi_i$  are Fourier coefficients of the unknown function  $\varphi(y)$ .

$$\bar{\Phi}_i = \sum_{j=1}^i A_{j,i} F_j.$$

It is well known that for an arbitrary  $\varepsilon$  there is an  $N_0$  such that when  $N > N_0$ , for any  $k < m$ , the following inequality will be satisfied

---

(14) Proof of linear independence and completeness of this system is analogous to that in the case of a singly connected region [3].

$$\int_{\Gamma_k} \left| \varphi(y) - \sum_{i=1}^N \Phi_i \varphi_i(y) \right|^2 ds_y < \varepsilon \quad (k=1, 2, \dots, m),$$

and a fortiori the approximate equation

$$\int_{\Gamma_k} \varphi(y) ds_y = \int_{\Gamma_k} \sum_{i=1}^N \Phi_i \varphi_i(y) ds_y \quad (k=1, 2, \dots, m).$$

/83

Therefore, taking into account (10.2) and (10.8) we find the following system for determining the constants  $C_k$ :

$$B_k = \sum_{i=1}^N \bar{\Phi}_i f_{i,k} = \sum_{i=1}^N 2\pi \sum_{j=1}^l C^{(j)} A_{j,i} f_{i,k} \quad (k=1, 2, \dots), \quad (10.9)$$

where

$$\int_{\Gamma_k} \varphi_i(y) ds_y = f_{i,k}.$$

Grouping in the right-hand side of the  $k^{\text{th}}$  equation ( $k = 1, 2, \dots, m$ ) (10.9) the terms with identical coefficient  $C_s$  ( $s = 1, 2, \dots, m$ ) and denoting their sum by  $A_{k,s}$ , we find the system

$$B_k = \sum_{s=1}^m \bar{A}_{k,s} C_s \quad (k=1, 2, \dots, m). \quad (10.10)$$

Determining from (10.10) the coefficients  $C_k$  ( $k = 1, 2, \dots, m$ ) and substituting them into (10.8) we find the Fourier coefficients of the unknown function  $\varphi(y)$ . Instead of  $\varphi(y)$  in (10.3) if we use the corresponding generalized Fourier series

$$\varphi^{(N)}(y) = \sum_{i=1}^N \Phi_i \varphi_i(y),$$

we find the approximate value of  $u^{(N)}(x)$  of the unknown solution to problem I at the arbitrary point  $x$  inside the region  $G$ . From the convergence of  $\varphi^{(N)}(y)$  to  $\varphi(y)$ , in the sense of the metric  $L_2$ , there directly follows the uniform convergence of  $u^{(N)}(x)$  to  $u(x)$ .

Substituting into (10.6) the integral of any quadrature formula and assigning to the parameter  $x$  different values of the auxiliary contour  $s$ , we find the following system:

$$\sum_{i=1}^N a_{i,j} \varphi_i = \sum_{i=1}^N \alpha_i \ln r(x_j, y_i) \varphi_i = F_j + 2\pi C^{(j)}, \quad (10.11)$$

where  $C^{(j)} = C_k$ , if  $x_j \in s_k$ . Let us write (10.11) in vector form

$$A\varphi = F + 2\pi C. \quad (10.12)$$

For the analyzed system we can prove the theorems which are analogous to Theorems 5 - 7 in Reference [4] (in Reference [4] in the formulation of Theorem 7 there is a misprint; instead of the asymptotic inequalities  $\lambda_1 \leq O(N^{-1/2})$ ,  $\lambda_1 \leq O(N^{-1})$ ,  $\mu_1 \leq O(N^{-1})$  and  $\|L^{-1}\| R_N \geq O(\sqrt{N})$ , respectively). Thus, for example, in complete analogy with the arguments cited in [3] we can prove that for any  $N$  there are  $N$  values  $x_k$  of the parameter  $x$  such that System (10.12) will be solvable, and the solution may be found by the method of successive approximations, beginning from the arbitrary vector  $\phi^{(0)}$ . In this case the norm in the sense of the space  $m_N$  of the inverse operator  $A^{-1}$  can be made as near to 1 as desired. From (10.12) we obtain

$$\varphi = A^{-1}F + 2\pi A^{-1}C. \quad (10.13)$$

Replacing the integral in (10.2) by the quadrature formula and using (10.13), we find a system of  $m$  equations for determining the coefficients  $C_k$  ( $k = 1, 2, \dots, m$ ). The values obtained for  $C_k$  are substituted into (10.13) and the vector  $\phi$  is determined, thus making it possible to compute the solution to the problem from Equation (10.3) by first replacing the integrals in it with quadrature sums.

For computation of the function  $v$ , we can use the algorithm from Reference [51], which assumes computation of the values  $v$  in the points at the limit by recurrence relationships (difference analog of the Cauchy-Riemann relationships), and in the interior points by solution of the system of difference equations. However, knowledge of the functions  $u|_{\Gamma}$  and  $\frac{\partial u}{\partial n}|_{\Gamma}$  makes it possible for computation of  $v$  to use the Green formula

$$v(x) = \frac{1}{2\pi} \int_{\Gamma} v(y) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y - \frac{1}{2\pi} \int_{\Gamma} \ln r(x, y) \frac{\partial v}{\partial n} ds_y,$$

where

$$v(y) = v_0 + \int_0^y \frac{\partial v}{\partial s} ds = v_0 + \int_0^y \frac{\partial u}{\partial n} ds = v_0 + \int_0^y \varphi(t) dt,$$

$$\frac{\partial v}{\partial n} = \frac{\partial u}{\partial s} = \omega'_s(s).$$

As far as Problems II and III are concerned, for their solution we can use the method of finite differences as described in [47]<sup>(15)</sup> or use the methods in References [1,3]. For the latter case we will give the formulas without discussing convergence of the respective computational algorithms.

Let us analyze Problem II. From its boundary values we determine

$$\frac{\partial u}{\partial n} \Big|_{\Gamma_k} = \varphi_k(s) - \gamma_k \left( \frac{\partial u}{\partial s} \right) \Big|_{\Gamma_k} \quad (k=p+1, \dots, m). \quad (10.14)$$

---

(15) However, we must note that the accuracy of the formulas in Reference ([47] for the points of the contours  $\Gamma_k$  ( $k=1, 2, \dots, m$ ) in the case of Problem II, and for the points of the boundary  $\Gamma$  in the case of Problem III, is by a whole order of magnitude inferior to the formulas in Reference [54]. Proofs of the convergence of this net-point method for Problems II and III in [47] are not vigorous, and the final computations are rather rough (compare with the computations in Reference [55]).

Using the Green formulas for Problem II and (10.14), we find

/85

$$\begin{aligned}
 u(x) = & \frac{1}{2\pi} \sum_{i=1}^p \int_{\Gamma_i} \psi(s) \frac{\partial}{\partial n_y} \ln r(x, y) ds - \\
 & - \frac{1}{2\pi} \sum_{k=p+1}^m \int_{\Gamma_k} \ln r(x, y) \left[ \varphi(s) - \gamma_k \frac{\partial u}{\partial s} \right] ds_y + \\
 & + \frac{1}{2\pi} \sum_{k=p+1}^m \int_{\Gamma_k} \bar{\psi}(s) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y - \\
 & - \frac{1}{2\pi} \sum_{i=0}^p \int_{\Gamma_i} \ln r(x, y) \varphi(s) ds_y, \quad x \in G_i,
 \end{aligned} \tag{10.15}$$

$$\begin{aligned}
 0 = & \sum_{i=0}^p \int_{\Gamma_i} \psi(s) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y - \sum_{k=p+1}^m \int_{\Gamma_k} \ln r(x, y) \left[ \varphi(s) - \right. \\
 & \left. - \gamma_k \frac{\partial u}{\partial s} \right] ds_y + \sum_{k=p+1}^m \int_{\Gamma_k} \bar{\psi}(s) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y - \\
 & - \sum_{i=0}^p \int_{\Gamma_i} \ln r(x, y) \bar{\varphi}(s) ds_y, \quad x \in G_e,
 \end{aligned} \tag{10.16}$$

where

$$\begin{aligned}
 \bar{\varphi}(s) &= \frac{\partial u}{\partial n} \Big|_{\Gamma_i} \quad (i=0, 1, \dots, p), \\
 \bar{\psi}(s) &= u \Big|_{\Gamma_k} \quad (k=p+1, \dots, m).
 \end{aligned}$$

Let us analyze the expression

$$\int_{\Gamma_k} \ln r(x, y) \frac{\partial u}{\partial s} ds \quad (k=p+1, \dots, m).$$

Applying to it the formula of integration by parts and taking into account the periodicity of the function  $\ln r(x, y)$  for variable  $y$ , we obtain

$$\int_{\Gamma_k} \ln r(x, y) \frac{\partial u}{\partial s} ds_y = - \int_{\Gamma_k} \bar{\psi}(s) \frac{\partial}{\partial s_y} \ln r(x, y) ds_y.$$

Taking this latter equation into account, (10.15) and (10.16) take the form

$$u(x) = \bar{F}(x) + \frac{1}{2\pi} \sum_{k=p+1}^m \int_{\Gamma_k} \bar{\psi}(s) \left[ \frac{\partial}{\partial n_y} \ln r(x, y) - \gamma_k \frac{\partial}{\partial s_y} \ln r(x, y) \right] ds_y -$$

$$- \frac{1}{2\pi} \sum_{i=0}^p \int_{\Gamma_i} \ln r(x, y) \bar{\varphi}(s) ds_y, \quad x \in G_i, \quad (10.17)$$

/86

$$0 = 2\pi \bar{F}(x) + \sum_{k=p+1}^m \int_{\Gamma_k} \bar{\psi}(s) \left[ \frac{\partial}{\partial n_y} \ln r(x, y) - \right.$$

$$\left. - \gamma_k \frac{\partial}{\partial s_y} \ln r(x, y) \right] ds_y - \sum_{i=0}^p \int_{\Gamma_i} \ln r(x, y) \bar{\varphi}(s) ds_y, \quad x \in G_e, \quad (10.18)$$

where

$$\bar{F}(x) = \frac{1}{2\pi} \sum_{i=0}^p \int_{\Gamma_i} \psi(s) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y -$$

$$- \frac{1}{2\pi} \sum_{k=p+1}^m \int_{\Gamma_k} \ln r(x, y) \varphi(s) ds_y$$

is a known function.

From the boundary conditions of Problem III we determine

$$\frac{\partial u}{\partial n} \Big|_{\Gamma_k} = \frac{1}{\gamma_k} \frac{\partial u}{\partial s} \Big|_{\Gamma_k} - \frac{\tau}{\gamma_k} \quad (k=p+1, \dots, m). \quad (10.19)$$

Applying the Green formula and (10.19) to solution of Problem III, we obtain

$$u(x) = \frac{1}{2\pi} \int_{\Gamma} \psi(s) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y +$$

$$+ \frac{1}{2\pi} \sum_{k=0}^p \int_{\Gamma_k} \ln r(x, y) \varphi(y) ds_y - \frac{1}{2\pi\gamma_k} \sum_{k=p+1}^m \int_{\Gamma_k} \left[ \frac{\partial u}{\partial s} - \right.$$

$$\left. - \varphi(y) \right] r(x, y) ds_y, \quad x \in G_i, \quad (10.20)$$

$$0 = \frac{1}{2\pi} \int_{\Gamma} \psi(s) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y + \frac{1}{2\pi} \sum_{k=0}^p \int_{\Gamma_k} \ln r(x, y) \varphi(y) ds_y -$$

$$- \frac{1}{2\pi\gamma_k} \sum_{k=p+1}^m \int_{\Gamma_k} \left[ \frac{\partial u}{\partial s} - \varphi(y) \right] r(x, y) ds_y, \quad x \in G_e, \quad (10.21)$$

where

$$\psi(s) = u \Big|_{\Gamma}.$$

Using the formula of integration by parts, Expressions (10.20) and (10.21) assume the following form:

$$u(x) = \frac{1}{2\pi} \sum_{k=0}^p \int_{\Gamma_k} \psi(s) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y + \frac{1}{2\pi} \int_{\Gamma} \ln r(x, y) \varphi(y) ds_y +$$

$$+ \frac{1}{2\pi} \sum_{k=p+1}^m \int_{\Gamma_k} \psi(y) \left[ \frac{\partial}{\partial n_y} \ln r(x, y) - \frac{1}{\gamma_k} \frac{\partial}{\partial s_y} \ln r(x, y) \right] ds_y, \quad x \in G, \quad (10.22) \quad /87$$

$$0 = \frac{1}{2\pi} \sum_{k=0}^p \int_{\Gamma_k} \psi(y) \frac{\partial}{\partial n_y} \ln r(x, y) ds_y + \frac{1}{2\pi} \int_{\Gamma} \ln r(x, y) \varphi(y) ds_y +$$

$$+ \frac{1}{2\pi} \sum_{k=p+1}^m \int_{\Gamma_k} \psi(y) \left[ \frac{\partial}{\partial n_y} \ln r(x, y) - \right.$$

$$\left. - \frac{1}{\gamma_k} \frac{\partial}{\partial s_y} \ln r(x, y) \right] ds_y, \quad x \in G. \quad (10.23)$$

Formulas (10.17) - (10.18) and (10.22) - (10.23) make it possible to use the methods described above for solving Problems II and III.

Let us indicate still one other method for solving the modified Dirichlet problem.

Let  $\{\varphi_i(x)\} [x(x_1, x_2)]$  be an orthonormalized system of functions harmonic in  $G_1$ . Furthermore, we shall assume that the system  $\{\varphi_i(x)\}$  is complete on  $\Gamma$  in

the sense of the metric  $L_2(\Gamma)$ , i.e., for any function  $\gamma(y) \in L_2(\Gamma)$  ( $y \in \Gamma$ ) and for any  $\varepsilon > 0$ , we find  $N$  of the coefficients  $b_i$  such that the following inequality will be satisfied

$$\left\{ \int_{\Gamma} \left[ \gamma(y) - \sum_{i=1}^N b_i \varphi_i(y) \right]^2 ds_y \right\}^{1/2} < \varepsilon.$$

Let us look at the boundary value problem (Figure 1)

$$\begin{aligned} \Delta u &= 0 & \text{in } G_i \\ u|_{\Gamma} &= \gamma(y). \end{aligned}$$

With the aid of the Schwarz inequality (and in the case of completeness in the sense of the metric  $C$  using the principle of the maximum) it is easy to show that in any interior point of the region  $G_i$  and uniformly in any region completely lying in the open region  $G_i$ , the difference

$$\left| u(x) - \sum_{i=1}^N b_i \varphi_i(y) \right|$$

can be made as small as desired. This method for solving the boundary value problems (the Piconet method [6]) may be quite effective with proper selection of the system  $\{\varphi(x)\}$ .

Let us look at the following problems:

$$\begin{aligned} \Delta u_0 &= 0 & \text{in } G, \\ u_0|_{\Gamma} &= \omega(s), \\ \Delta u_i &= 0 & \text{in } G_i, \end{aligned} \tag{10.23_1}$$

$$\begin{aligned} u_i|_{\Gamma - \Gamma_i} &= 0 & k = (1, 2, \dots, m), \\ u_i|_{\Gamma_i} &= 1. \end{aligned}$$

/88

Let the approximate solutions to these problems be represented in the form of the following series:

$$u_i \approx \sum_{j=1}^{N_i} A_{j,i} \varphi_j = \bar{u}_i \quad (i=0, 1, 2, \dots, m).$$

Let us introduce the following notations

$$\int_{\Gamma_k} \frac{\partial \bar{u}_i}{\partial n} ds = e_{k,i} \quad (i=0, 1, 2, \dots, m; \quad k=1, \dots, m).$$

To determine the coefficients  $c_i$  ( $i = 1, \dots, m$ ) of Problem (10.1) - (10.2), we obtain the following system:

$$\sum_{i=1}^m c_i e_{k,i} = e_{k,0}, \quad k=1, 2, \dots, m. \quad (10.3_2)$$

Determining  $c_i$  from this latter system, the approximate solution  $\bar{u}$  to Problem (10.1) - (10.2) takes the following form:

$$\bar{u} = \sum_{i=0}^N c_i \bar{u}_i,$$

where  $c_0 = 0$ .

Usually [6] in the case of a singly connected region, as such a system we take the system  $\{p_i(x)\}$  of harmonic polynomials; we know that they are complete in  $L_2(\Gamma)$ , where  $\Gamma$  is a contour bounding a singly connected region (if this region has a stable solution of the Dirichlet problem with respect to deformation of the region, then the system of harmonic polynomials is complete on  $\Gamma$  in the sense of the metric  $C$ ).

It is clear then in the case of a multiply connected region the system  $\{p_i(x)\}$  can be complete neither in the sense of the metric  $C$  nor in the sense of the metric  $L_2$ .

For any continuous function  $\gamma(y)$  let

$$\max_{y \in \Gamma} \left| \gamma(y) - \sum_{i=1}^N b_i p_i(y) \right| < \varepsilon \quad (y \in \Gamma).$$

From the principle of the maximum it directly follows that if for the function  $\gamma(y)$  the following inequality is satisfied

$$\max_{y \in \Gamma} |\gamma(y)| > \max_{y \in \Gamma_0} |\gamma(y)|,$$

then such a function cannot be approximated sufficiently well by harmonic polynomials. However, this latter inequality may not be satisfied for the function  $\gamma(y)$  either, and cannot be approximated by harmonic polynomials.

In fact let us take two functions  $\gamma_1(y)$  and  $\gamma_2(y)$ , for which the following /89  
equations are satisfied:

$$\begin{aligned} \gamma_1(y) &= \gamma_2(y) \quad \text{on } \Gamma_0 \quad (\text{Figure 1}) \\ \max_{y \in \Gamma - \Gamma_0} |\gamma_1(y) - \gamma_2(y)| &= 1. \end{aligned} \tag{10.24}$$

On the strength of our assumption for any  $\varepsilon > 0$  we find coefficients  $b_{\lambda}^{(1)}$  and  $b_i^{(2)}$  such that

$$\max_{y \in \Gamma} \left| \gamma_1 - \sum_{i=1}^{N_1} b_i^{(1)} p_i(y) \right| < \frac{\varepsilon}{4}, \quad \max_{x \in \Gamma} \left| \gamma_2 - \sum_{i=1}^{N_2} b_i^{(2)} p_i(y) \right| < \frac{\varepsilon}{4}.$$

From the principle of the maximum and the first of the equations in (10.24), we find that at any point of the region  $G$  bounded by the contour  $\Gamma_0$ , the following inequality is valid

$$\left| \sum_{i=1}^{N_1} b_i^{(1)} p_i(y) - \sum_{i=1}^{N_2} b_i^{(2)} p_i(y) \right| < \frac{\varepsilon}{2},$$

and therefore,

$$\begin{aligned} \max_{y \in \Gamma - \Gamma_0} \left| \gamma_1(y) - \gamma_2(y) \right| &< \max_{y \in \Gamma - \Gamma_0} \left| \gamma_1(y) - \sum_{i=1}^{N_1} b_i^{(1)} p_i(y) + \right. \\ &\left. + \sum_{i=1}^{N_2} b_i^{(2)} p_i(y) - \gamma_2(y) \right| + \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

which contradicts the second equation in (10.24).

The system  $\{p_i(x)\}$  may not be complete in the sense of  $L_2(\Gamma)$  either. Let us look at the problem

$$\begin{aligned} \Delta u &= 0 \text{ on } G', \\ u \Big|_{\Gamma_0} &= \gamma_1(y), \end{aligned}$$

and the function  $\gamma(y)$ , which satisfies the conditions

$$\begin{aligned} \gamma(y) &= \gamma_1(y) \text{ на } \Gamma_0, \\ \gamma(y) &= u(y) + C_k \text{ на } \Gamma_k \quad (k=1, \dots, m), \end{aligned}$$

where  $u(y) = u(x^k) \Big|_{\Gamma}$  and at least one of the constants  $C_k$  is not zero.

From the representation of the solution to the Dirichlet problem for the region  $G$  with the aid of the Green function and from the Schwarz inequality, it becomes clear that it is impossible to approximate the function  $\gamma(y)$  sufficiently well in the sense of the metric  $L_2$  by harmonic polynomials. From these arguments it follows that the functions from the complete (either in the sense of the metric  $L_2(\Gamma)$ , or the sense of the metric  $C(\Gamma)$ ) system cannot all be harmonic simultaneously at all points of the region  $G'$ .

As far as the author knows, the only system of harmonic functions complete /90 in the sense of the metric  $L_2(\Gamma)$  is given in Reference [3].

$$\{\ln r(x_i, x)\}, \quad (10.25)$$

where the points  $x_i$  are distributed everywhere densely on  $s = s_0 + s_1 + \dots + s_m$  (Figure 3).

We must also mention that this system greatly facilitates computation of the coefficients  $e_{k,i}$  ( $k = 1, 2, \dots, m$ ;  $i = 0, 1, \dots, m$ ) of the system from which  $c_i$  ( $i = 1, 2, \dots, m$ ) are determined.

In fact it is easy to see that

$$e_{h,i} = 2\pi \sum_{j=1}^{N_i} r_j A_{j,i},$$

where

$$u_i = \sum_{j=1}^{N_i} A_{j,i} \ln r(x_j, x), \quad r_i = \begin{cases} 1, & x_j \in G_h \\ 0, & x_j \in G_e - G_h. \end{cases}$$

Analogously the computations of the coefficients are simplified if instead of System (10.25) we analyze the corresponding orthonormalized system.

In conclusion let us prove the solvability of System (10.10) and (10.3<sub>2</sub>). It is easy to see that  $\bar{A}_{k,s}$  and  $B_k$  ( $k, s = 1, 2, \dots, m$ ) in (10.10), when  $N \rightarrow \infty$  approach the expressions

$$\int_{\Gamma_h} \frac{\partial u_s}{\partial n} ds, \quad \int_{\Gamma_h} \frac{\partial u_0}{\partial n} ds,$$

respectively, where  $u_s$  are determined in (10.23<sub>1</sub>).

The coefficients  $e_{k,i}$  and  $e_{k,0}$  in (10.3<sub>2</sub>) approach these same expressions under the condition

$$\lim_{N \rightarrow \infty} \left\| \frac{\partial}{\partial n} (u_i - \bar{u}_i) \right\|_{L_2} = 0 \quad (i=0, 1, \dots, m),$$

so that the limiting form of Systems (10.10) and (10.23<sub>2</sub>) coincides. Therefore, further discussions on the solvability of these systems are completely analogous to those given on pages 98 and 100. We shall omit these and refer the reader to those pages.

#### §11. Solution to the External Dirichlet Problem for the Laplace Equation Using the Method of Functional Equations.

The program is intended for obtaining a harmonic function outside the ellipse

$$s : y_1 = \cos \alpha, \quad y_2 = b \sin \alpha$$

under the boundary condition  $u|_s = f(y), y \in s$ .

The program consists of four parts:

/91

I. An operational program for transferring from the third and fourth parts from the main memory to the drum, and also for transferring the initial data to the main memory and for transferring the third and fourth parts from the drum to the main memory.

II. A program which converts the array of coordinate points outside the ellipse from the decimal notation system into a binary one and transfers them to the drum.

III. A program which computes the axes of the auxiliary boundary  $s_1$ , as well as the coordinates of the points on this boundary and prints out these coordinates. It also computes the scalar products  $(\omega_k, \omega_i)$ , and the coefficients of orthonormalization  $A_{k,i}$ ; it computes the orthonormalized functions and verifies the orthonormalization.

IV. A program for computing the values of the harmonic function at given points outside the ellipse. The program prints out the coordinates of these points and the solutions corresponding to them.

#### Computational Procedure

The computational process consists of the following stages:

1. Computation of the coordinates of the auxiliary points and scalar products.
2. Computation of the coefficients of orthonormalization.
3. Computation of the orthonormalized functions and proof of orthonormalization.
4. Computation of the Fourier coefficients and the normal derivative of the unknown function.
5. Computation of the values for solution to the problem.

# 1. Computation of the Coordinates of the Auxiliary Points and Scalar Products.

The auxiliary points are taken on the confocal ellipse with semiaxes  $\bar{a} = \frac{\bar{b}}{h}$  and  $\bar{b} = b - (h - \bar{b})$ , where  $(b - \bar{b})$  is given in the initial data (cell 0622). The coordinates of the auxiliary points are computed from the formula

$$x_1^{(k)} = \bar{a} \cos \alpha_k, \quad x_2^{(k)} = \bar{b} \sin \alpha_k \quad (k=1, 2, \dots, 24). \quad (1.1)$$

$\alpha_k$  ( $k=1, 2, \dots, 24$ ) assume the values given in Table 7.

The scalar products

$$(\omega_h, \omega_l) = \int_0^{2\pi} \omega_h(\alpha) \omega_l(\alpha) d\alpha, \quad (1.2)$$

where

$$\omega_h(\alpha) = \frac{1}{2} \ln \left[ (\cos \alpha - x_1^{(h)})^2 + (b \sin \alpha - x_2^{(h)})^2 \right]$$

are computed by Simpson's method.

# 2. Computation of the Coefficients of Orthonormalization.

The coefficients of orthonormalization  $A_{k,i}$  are computed from the following formulas:

$$A_{k,k} = \frac{1}{\|g_k\|} \quad (k=1, 2, \dots, 24), \quad (2.1)$$

$$A_{k,h-i} = A_{k,h} \sum_{j=1}^i \alpha_{h,j} A_{k-j,h-i} \quad (k=2, 3, \dots, 24; i=1, 2, \dots, 23), \quad (2.2)$$

where

$$\|g_k\| = \sqrt{\sum_{j=0}^{k-1} B_{k, k-j}^2 \int_0^{2\pi} \omega_{k-j}^2 d\alpha + 2 \sum_{j=0}^{k-2} B_{k, k-j} \sum_{i=1}^{k-j-1} B_{k, i} \int_0^{2\pi} \omega_{k-j} \omega_i d\alpha}, \quad B_{k, k} = 1, \quad (2.3)$$

$$\alpha_{k, k-j} = - \sum_{i=1}^{k-j} A_{k-j, i} \int_0^{2\pi} \omega_k \omega_i d\alpha \quad (k=2, 3, \dots, 24; j=k-1, k-2, \dots, 1), \quad (2.4)$$

$$B_{k, j} = \sum_{i=1}^{k-j} \alpha_{k, k-j} A_{k-i, j}, \quad (k=2, 3, \dots, 24; j=k-1, k-2, \dots, 1). \quad (2.5)$$

### 3. Computation of the Orthonormalized Functions and Checking of Orthonormalization.

The value of the orthonormalized functions is computed at the points  $\alpha_j = (j-1) \frac{\pi}{125}$  by the formula

$$\varphi_k(\alpha_j) = \sum_{i=1}^k A_{k, i} \omega_i(\alpha_j), \quad (k=1, 2, \dots, 24; j=1, 2, \dots, 250). \quad (3.1)$$

The correctness of the orthonormalization is checked by the formula

$$1 - (\varphi_k, \varphi_i) = 1 - \frac{\pi}{125} \sum_{j=1}^{250} \varphi_k(\alpha_j) \varphi_i(\alpha_j) < 10^{-4} \quad \text{when } k=i,$$

$$(\varphi_k, \varphi_i) = \frac{\pi}{125} \sum_{j=1}^{250} \varphi_k(\alpha_j) \varphi_i(\alpha_j) < 10^{-4} \quad \text{when } k \neq i.$$

### 4. Computation of the Fourier Coefficients and the Normal Derivative of 93 the Unknown Function.

The Fourier coefficients of the unknown function  $\varphi(y)$  are computed by the formula

$$\Phi_i = \sum_{k=1}^i A_{k,i} F_k \quad (i=1, 2, \dots, 24), \quad (4.1)$$

where

$$F_k = \int_0^{2\pi} f(\alpha) \frac{b - (b \cos \alpha x_1^{(k)} + \sin \alpha x_2^{(k)})}{(\cos \alpha - x_1^{(k)})^2 + (b \sin \alpha - x_2^{(k)})^2} d\alpha, \quad (k=1, 2, \dots, 24) \quad (4.2)$$

$A_{k,i}$  are the coefficients of orthonormalization.

The normal derivative of the unknown function is computed at the points  $\alpha_j = (j-1) \frac{\pi}{125}$  by the formula

$$\varphi(\alpha_j) = \sum_{i=1}^{24} \Phi_i \varphi_i(\alpha_j) \quad (j=1, 2, \dots, 250) \quad (4.3)$$

## 5. Computation of the Values for Solution to the Problem.

The value for the solution to the problem at points outside the ellipse is sought with the formula

$$u(\xi, \eta) = \frac{1}{250} \sum_{j=1}^{250} \left\{ -f(\alpha_j) \frac{b - (b \cos \alpha_j \xi + \sin \alpha_j \eta)}{(\cos \alpha_j - \xi)^2 + (b \sin \alpha_j - \eta)^2} + \right. \quad (5.1)$$

$$\left. + \frac{1}{2} \ln \left[ (\cos \alpha_j - \xi)^2 + (b \sin \alpha_j - \eta)^2 \right] \varphi(\alpha_j) \right\}, \quad \alpha_j = (j-1) \frac{\pi}{125}, \quad (\xi, \eta) \in B_a. \quad (5.1)$$

## FLOW DIAGRAM

### Overall Flow Diagram

- A Operating program.
- B Standard subprogram block.
- C Computation of coordinates of points outside ellipse.
- D Computation of coordinates of auxiliary points and scalar products.
- E Computation of coefficients of orthonormalization.
- F Computation of orthonormalized functions and proof of orthonormalization.
- G Computation of Fourier coefficients and the normal derivative of the unknown function.
- H Computation of the value for solution to the problem.

### Block A

/94

- A 1 Read in fundamental program.
- A 2 Transfer third and fourth parts of program to drum.
- A 3 Read in input card for initial data.
- A 4 Read in initial data.
- A 5 Refer to B 1.
- A 6 Refer to C.
- A 7 Transfer third part of program to main memory from drum and access D.
- A 8 Transfer fourth part of program to main memory from drum and access G.

### Block B

- B 1 Block transfer array " $10 \rightarrow 2$ ".
- B 2 Compute  $\sin x$  and  $\cos x$ .
- B 3 Compute  $\log x$ .

- B 4 Block transfer " $2 \rightarrow 10$ ".
- B 5 Compute specified integral by Simpson's method with automatic calling sequence (Authors L. S. Tsyganova, I. L. Klimkina).
- B 6 Square root extraction.

#### Block C

- C 1 Transfer initial data to standard cells.
- C 2 Transfer programs for computing boundary function  $f(\alpha)$  onto drum from main memory.
- C 3 Check the number of coordinates in the memory of the machine at the same time.
- C 4 Check how many points are given outside ellipse.
- C 5 Set up shaping constants.
- C 6 Shape commands for input of numerical data from reader to main memory.
- C 7 Read in numerical data.
- C 8 Shape commands for block transfer " $10 \rightarrow 2$ ".
- C 9 Refer to B 1.
- C 10 Shape commands for transfer to drum.
- C 11 Transfer numerical data to drum.
- C 12 Check if all numerical data have been fed into machine; if yes go to A 7, if no go to C 13.
- C 13 Readdress shaping constants and go to C 6.

#### Block D

- D 1 Compute  $\bar{a}$  and  $\bar{b}$ .
- D 2 Clear counter for  $k$ .
- D 3 Compute coordinates of auxiliary points from Formula (1.1).
- D 4 Refer to B 4.
- D 5 Print out  $x_1^{(k)}$  and  $x_2^{(k)}$ .
- D 6 Readdress variable commands.
- D 7 Check if coordinates of all auxiliary points have been computed; if yes go to D 8, if no go to D 3.
- D 8 Set up variable commands for computation from Formula (1.2).

/95

- D 9 Refer to B 5.
- D 10 Transfer  $(\omega_k, \omega_i)$  to storage.
- D 11 Readdress variable commands.
- D 12 Check if all  $(\omega_k, \omega_i)$  have been computed; if yes go to E; if no go to D 9.

### Block E

- E 1 Transfer  $(\omega_1 \omega_1)$  to cell 0001.
- E 2 Refer to B 6.
- E 3 Compute  $A_{1.1} = \frac{1}{\sqrt{(\omega_1 \omega_1)}}$ .
- E 4 Transfer constants (00 0002 0000 0000) into counter for k.
- E 5 Retrieve variable commands and set up constants for shaping.
- E 6 Clear counter for j.
- E 7 Shape commands for computing products  $A_{k-j,i} (\omega_k \omega_i)$ .
- E 8 Compute products  $A_{k-j,i} (\omega_k \omega_i)$ .
- E 9 Compute Formula (2.4).
- E 10 Check if all terms in Formula (2.4) have been computed; if yes go to E 11; if no go to E 14.
- E 11 Transfer values  $\alpha_k, k-j$  to storage.
- E 12 Readdress variable commands and constants.
- E 13 Check if all  $\alpha_k, k-j$  have been computed; if yes go to E 15; if no go to E 5.
- E 14 Set up shaper constants for computing the next term and go to E 7.
- E 15 Set up constants for shaping.
- E 16 Clear counter for j.
- E 17 Shape commands for computing the product  $\alpha_{k,k-i} A_{k-i,j}$ .
- E 18 Compute product  $\alpha_{k,k-i} A_{k-i,j}$ .
- E 19 Compute Formula (2.5).
- E 20 Check if all terms  $B_{k,j}$  have been computed; if yes go to E 21; if no go to E 24.
- E 21 Transfer values  $B_{k,j}$  to storage.
- E 22 Readdress variable commands and constants.

E 23 Check if all  $B_{k,j}$  have been computed; if yes go to E 25, if no go to E 17.

E 24 Set up shaping constants for computing the next term and go to E 17.

/96

E 25 Set up constants for shaping.

E 26 Shape commands for computing the formula

$$\sum_{j=0}^{k-1} B_{k, k-j}^2 \int_0^{2\pi} \omega_{k-j}^2 d\alpha. \quad (6)$$

E 27 Compute product  $B_{k, k-j}^2 \int_0^{2\pi} \omega_{k-j}^2 d\alpha$ .

E 28 Compute Formula (6).

E 29 Check if all terms in Formula (6) have been computed; if yes go to E 31; if no go to E 30.

E 30 Readdress variable commands for computing next term and go to E 27.

E 31 Shape commands for computing the formula

$$2 \sum_{j=0}^{k-2} B_{k, k-j} \sum_{i=1}^{k-j-1} B_{k, i} \int_0^{2\pi} \omega_{k-j} \omega_i d\alpha. \quad (7)$$

E 32 Compute product  $B_{k, k-j} B_{k, i} \int_0^{2\pi} \omega_{k-j} \omega_i d\alpha$ .

E 33 Compute Formula (7).

E 34 Check if all terms in Formula (7) have been computed; if yes go to E 36; if no go to E 35.

E 35 Readdress variable commands for computing next term and go to E 32.

E 36 Compute radicand.

E 37 Refer to B 6.

E 38 Shape commands for computing Formula (2.1).

E 39 Compute Formula (2.1).

E 40 Set up constants for shaping.

- E 41 Shape commands for computing Formula (2.2).
- E 42 Transfer constants (00 0001 0000 0000) to i-counter.
- E 43 Compute nondiagonal coefficient of given row using Formula (2.2).
- E 44 Check if all nondiagonal coefficients of given row have been computed; if yes go to E 46; if no go to E 45.
- E 45 Readdress commands for computing next nondiagonal coefficient and go to E 43.
- E 46 Check if all coefficients  $A_{k,i}$  have been computed; if yes go to F; if not go to E 5.

#### Block F

/97

- F 1 Retrieve variable commands.
- F 2 Clear counter for k.
- F 3 Compute Formula (3.1).
- F 4 Check if  $\varphi_k$  have been computed for all values  $\alpha_j$ ; if yes go to F 5, if no go to F 3.
- F 5 Transfer computed  $\varphi_k(\alpha_j)$  to drum.
- F 6 Readdress variable commands.
- F 7 Check if all  $\varphi_k$  have been computed; if yes go to F 8; if no go to F 3.
- F 8 Retrieve variable commands and set up constants.
- F 9 Copy values  $\varphi_k(\alpha_j)$  from drum.
- F 10 Compute the formula

$$c = \frac{\pi}{125} \sum_{j=1}^{250} \varphi_k(\alpha_j) \varphi_l(\alpha_j).$$

- F 11 Check  $k = i$ ; if yes go to F 17; if no go to F 12.
- F 12 Check Condition (3.3); if it is satisfied go to B 21; if not go to F 13.
- F 13 Transfer value C to cell 0001.
- F 14 Refer to B 4.
- F 15 Print out C.
- F 16 Refer to F 21.

F 17 Check Condition (3.2); if satisfied go to B 21; if not go to F 18.  
 F 18 Transfer value  $1 - C$  to cell 0001.  
 F 19 Refer to B 4.  
 F 20 Print out  $1 - C$ .  
 F 21 Readdress variable commands and constants.  
 F 22 Check if Conditions (3.2) and (3.3) have been verified for all  $\varphi_k$ ; if  
       yes go to A 8, if no go to F 9.

### Block G

G 1 Transfer programs for computing boundary function  $f(\alpha)$  from drum to main  
       memory.  
 G 2 Clear counter for  $k$ .  
 G 3 Transfer coordinates of auxiliary points  $x_1^{(k)}$  and  $x_2^{(k)}$  to working cells.  
 G 4 Refer to B 5.  
 G 5 Readdress variable commands.  
 G 6 Check if all  $F_k$  have been computed; if yes go to G 7; if no go to G 3.  
 G 7 Set up variable commands and shaping constants.  
 G 8 Clear counter for  $i$ .  
 G 9 Shape commands for computing product  $A_{k,i} F_k$ . /98  
 G 10 Compute product  $A_{k,i} F_k$ .  
 G 11 Compute Formula (4.1).  
 G 12 Check if all terms of Formula (4.1) have been computed; if yes go to  
       G 13; if no go to G 10.  
 G 13 Transfer values  $\phi_i$  to storage.  
 G 14 Readdress variable commands and constants.  
 G 15 Check if all  $\phi_i$  have been computed; if yes go to G 16; if no go to G 9.  
 G 16 Retrieve variable commands and set up constants for shaping.  
 G 17 Clear counter for  $i$ .  
 G 18 Transfer value  $\phi_i(\alpha_j)$  from drum to main memory.  
 G 19 Shape commands for computing Formula (4.3).  
 G 20 Compute Formula (4.3).  
 G 21 Readdress variable commands.

G 22 Check if values of normal derivative of unknown function  $\varphi(\alpha)$  for all  $\alpha_j$  have been computed; if yes go to G 23; if no go to G 18.  
 G 23 Check if necessary to print out; if yes go to G 24; if no go to H.  
 G 24 Refer to B 4.  
 G 25 Print out  $\varphi(\alpha_j)$ .

#### Block H

H 1 Set up shaping constants  
 H 2 Transfer coordinates of points outside ellipse from drum to main memory.  
 H 3 Retrieve variable commands and constants.  
 H 4 Compute Formula (5.1).  
 H 5 Refer to B 4.  
 H 6 Print out  $\xi$ ,  $\eta$  and  $u(\xi, \eta)$ .  
 H 7 Readdress variable commands.  
 H 8 Check if computation of Formula (5.1) for all transferred points is finished; if yes go to H 9; if no go to H 4.  
 H 9 Check if all points are transferred from drum; if yes to go to H 11, if no go to H 10.  
 H 10 Readdress shaping constants and go to H 2.  
 H 11 Stop.

#### INSTRUCTION

##### 1. Set Up of Initial Data

The initial data for the program consist of two parts:

1. Initial data in which programs are included for computing the boundary function and certain initial data.

The program for computing the boundary function  $f(\alpha)$  must begin with cell 0626. The argument  $\alpha$  is taken from cell 0001, and the result must be

/99

transferred to cell 0001. In the operation of the program for the boundary function in cells 0017 and 0020, we will find  $\cos \alpha$  and  $b \sin \alpha$ , respectively. They may be used for computing the boundary function. In certain instances this greatly reduces the commands in the program for computing the boundary function.

The program for computing the boundary function certainly must end with the command 34 - - 0555, i.e., after its operation it must transfer control to cell 0555. In setting up the program for computing the boundary function, as the working cells we can use 0013-0016, 0021-0033 and 0616-0625.

The following initial data are transferred to cells 0622-0625:

0622-( $b-\bar{b}$ ) is the difference between the minor semiaxes of the fundamental S and the auxiliary  $S_1$  ellipses in the decimal notation system.

0623-b is the minor semiaxis of the basic ellipse, in the decimal notation system.

0623-N is the length of the program for computing the boundary function  $f(\alpha)$  in the third address in the octal notation system.  $N < 600_{(10)}$ .

0625 is the cell with conditional data. Depending on which of the two numbers 0000, 0010 is found in the first address, the program will transfer the following numbers: 0010 - 250 values are printed out for the normal derivative of the unknown function at the points  $\alpha_j = (j-1) \frac{\pi}{125}$ . 0000 - no values are printed out for the normal derivative of the unknown function.

In the second address  $N_1$  is registered - the number of points at which we are required to find a solution in the octal system of notation  $N_1 < 3777_{(8)}$ , and in the third address - 0000.

2. Numerical data (coordinates  $(\xi_i, \eta_i)$  of the points outside the

ellipse) must follow directly after the program for computing the boundary function. Thus, in cell  $0626 + N$  we must find the abscissa  $\xi_1$  of point  $Q_1$ . In cell  $0627 + N$  — the ordinate  $\eta_1$  of this point. In the cell  $0630 + N$  — the abscissa  $\xi_2$  of point  $Q_2$ , etc.

For input of the initial data we must make the following "input card":

30 0100	—	$n-1$
31	0622	0010
77		0060
0622	0002	0622
34		0134

where  $n$  is the length of the initial data.

## 2. Operation of Console

/100

The entire punchcard deck is assembled in the following order:

1. Operation code.
2. Blank card.
3. Initial data input card.
4. Blank card.
5. Initial data.
6. Blank card.
7. Numerical data.

TABLE OF CONTROL STOPS

Stop	Contents of instruction storage				Reason for stop	Action
0004	33	0001	—	—	Incorrect program input.	Repeat read in.
0013	33	—	0001	—		
0022	33	—	0002	—	Incorrect drum access.	With key start repeat access to drum.
0032	33	—	0003	—		
0037	33	—	0004	—		
0151	33	—	0005	—		
0216	33	—	0006	—		
1036	33	—	0007	—	Incorrect conversion of numbers from decimal to binary system.	Clear problem and enter again.
1056	33	—	0010	—		
1064	33	—	0011	—		
0344	33	—	0012	—		
0416	33	—	0013	—		
0464	33	—	0014	—		
0073	33	—	0002	—		
0153	33	—	0013	—	Argument for which logarithm is sought is negative.	
0365	33	—	0010	—	Radicand negative.	Carry out command 34 - - 0570 from console.
1104	33	—	—	0001	Product $(\varphi_k, \varphi_i)$ when $k \neq i$ is greater than $10^{-4}$ .	Continue computation with key start.
1114	33	—	—	0002	Difference $1 - (\varphi_k, \varphi_i)$ when $k = i$ is greater than $10^{-4}$ .	Continue computation with key start.
0533	33	0002	—	—	End of computation.	

If the stops given above are repeated several times, then the problem must be cleared and the machine checked.

It may happen that with the machine in good working order there occurs the stop 0365 (33 - 0010 -). In this case the difference  $(b - \bar{b})$  must be decreased.

If the machine is in good working order and if the stops 1104 (33 — 0001) and 1114 (33 - - 0002) are repeated several times, then it is necessary to decrease the difference  $(b - \bar{b})$ .

/101

### Interpretation of the Final Results of Machine Print Out

The program prints out the coordinates of the auxiliary points in the sequence:

$$x_1^{(1)}, x_2^{(1)}; x_1^{(2)}, x_2^{(2)}; x_1^{(3)}, x_2^{(3)}; \dots, x_1^{(24)}, x_2^{(24)}.$$

In checking the orthonormalization it may happen that Conditions (3.2) and (3.3) are not satisfied. If Condition (3.2) is not satisfied, then the machine prints out the value of the difference  $1 - (\varphi_k, \varphi_l)$  and the conditional digit (00 000 001) and if Condition (3.3) is not satisfied then only the value  $(\varphi_k, \varphi_l)$  is printed.

If it is required to print out the values of the normal unknown function  $\varphi(\alpha)$ , then the program prints out 250 values at the points  $\alpha_j = (j-1) \frac{\pi}{125}$  ( $j = 1, 2, \dots, 250$ ) in the sequence:

$$\varphi(\alpha_1), \varphi(\alpha_2), \dots, \varphi(\alpha_{250}).$$

The program prints out the coordinates of the points at which solution to the boundary value problem and the solutions corresponding to it are found

$$\xi_1, \eta_1, u(\xi_1, \eta_1); \xi_2, \eta_2, u(\xi_2, \eta_2); \dots, \xi_{N_1}, \eta_{N_1}, u(\xi_{N_1}, \eta_{N_1}).$$

### Numerical Examples

Numerical Example 1. Let us look at the solution to the Dirichlet

problem for the ellipse

$$S: y_1 = \cos \alpha, \quad y_2 = 0,80 \sin \alpha,$$

under the boundary condition

$$f(y) = \frac{\xi}{\xi^2 + \eta^2}, \quad y(\xi, \eta) \in S.$$

The auxiliary points  $x^{(k)} \in B_i$  were taken on the confocal ellipse

$$S_1: x_1^{(k)} = 0,75 \cos \alpha_k, \quad x_2^{(k)} = 0,60 \sin \alpha_k \quad (k=1, 2, \dots, 24).$$

Such values for the semiaxes of the auxiliary ellipse of the program are selected in the case when as the initial data in the cell 0622 we find the number 0.20.

The coordinates of the auxiliary points

$$x_1^{(1)}, x_2^{(1)}; x_1^{(2)}, x_2^{(2)}, \dots, x_1^{(24)}, x_2^{(24)}$$

on the ellipse  $S_1$  are given in Table 8.

In this case, it was found that the orthonormalization was checked with /102 an accuracy exceeding  $10^{-4}$ , i.e., the diagonal terms within an accuracy of  $10^{-4}$  were equal to unity and the nondiagonal terms within this same accuracy were equal to zero.

Since as the initial data we were not required to print out the values for the normal derivative of the unknown function  $\varphi(\alpha_i)$  (in the first address of cell 0625 we have 0000), the program does not print them out.

A precise solution to this problem has the form

$$u(\xi, \eta) = \frac{\xi}{\xi^2 + \eta^2}, \quad (\xi, \eta) \in B_a.$$

and, consequently, it was possible to compare the approximate solution with the exact solution and compute the error  $\varepsilon$ .

Table 9 gives the coordinates of the points outside the ellipse, the approximate solutions to them and the deviation of the approximate solution from the exact one.

Numerical Example 2. As another numerical example, let us look at the solution to the Dirichlet problem for the ellipse,

$$S : y_1 = \cos \alpha, \quad y_2 = 0,50 \sin \alpha,$$

under the boundary condition

$$f(y) = -\frac{2\xi\eta}{(\xi^2 + \eta^2)^2}, \quad y(\xi, \eta) \in S.$$

The auxiliary points  $x^{(k)} \in B_l$  were taken on the conformal ellipse

$$S_1 : x_1^{(k)} = 0,80 \cos \alpha_k, \quad x_2^{(k)} = 0,50 \sin \alpha_k \quad (k = 1, 2, \dots, 24).$$

The coordinates of the auxiliary points are given in Table 10.

In this case, the orthonormalization was also carried out with an accuracy exceeding  $10^{-4}$ .

The exact solution to this problem has the form

$$u(\xi, \eta) = -\frac{2\xi\eta}{(\xi^2 + \eta^2)^2}, \quad (\xi, \eta) \in B_a.$$

Table 11 gives the coordinates of the points outside the ellipse, the approximate solutions corresponding to them and the deviation of the approximate solution from the exact one.

The distribution of points outside the ellipse, at which the values are computed for solution to the boundary value problem, for the first numerical example is given on Figure 4, and for the second example - on Figure 5.

TABLE 7

$k$	$\alpha_k$	$k$	$\alpha_k$	$k$	$\alpha_k$
1	90°	9	330°	17	75°
2	270°	10	150°	18	255°
3	0°	11	120°	19	105°
4	180°	12	300°	20	285°
5	225°	13	60°	21	15°
6	45°	14	240°	22	195°
7	315°	15	30°	23	165°
8	135°	16	210°	24	345°

TABLE 8

$k$	$x_1^{(k)}$	$x_2^{(k)}$	$k$	$x_1^{(k)}$	$x_2^{(k)}$
1	-0.10971885 · 10 <sup>-7</sup>	0.60000000	13	0.37499999	0.51961524
2	0.32915659 · 10	-0.60000000	14	-0.37499997	-0.51961525
3	0.75000000	0.00000000	15	0.64951905	0.30000000
4	-0.75000000	-0.17555016 · 10 <sup>-7</sup>	16	-0.64951904	-0.30000001
5	-0.53033006	-0.42426408	17	0.19411427	0.57955549
6	0.53033008	0.42426407	18	-0.19411425	-0.57955550
7	0.53033011	-0.42426404	19	-0.19411429	0.57955549
8	-0.53033009	0.42426406	20	0.19411431	-0.57955549
9	0.64951907	-0.29999997	21	0.72444437	0.15529142
10	-0.64951906	0.29999997	22	-0.72444436	-0.15529144
11	-0.37500001	0.51961523	23	-0.72444437	0.15529141
12	0.37500003	-0.51961522	24	0.72444438	-0.15529139

/103

TABLE 9

$N_1$	$\xi_{N_1}$	$\eta_{N_1}$	$u(\xi_{N_1}, \eta_{N_1})$	$\varepsilon$
1	0.80	1.00	0.48780488	-0.1 · 10 <sup>-7</sup>
2	0.80	1.20	0.38461536	0.2 · 10 <sup>-7</sup>
3	0.80	1.40	0.30769227	0.3 · 10 <sup>-7</sup>
4	0.80	1.60	0.24999996	0.4 · 10 <sup>-7</sup>
5	0.80	1.80	0.20618553	0.3 · 10 <sup>-7</sup>
6	0.80	2.00	0.17241376	0.3 · 10 <sup>-7</sup>
7	0.80	2.40	0.12499997	0.3 · 10 <sup>-7</sup>
8	0.80	2.60	0.10810808	0.2 · 10 <sup>-7</sup>
9	0.90	1.10	0.44554454	0.1 · 10 <sup>-7</sup>

TABLE 9 (con't)

10	0.90	1.60	0.26706228	$0.3 \cdot 10^{-7}$
11	0.90	2.10	0.17241376	$0.3 \cdot 10^{-7}$
12	0.90	2.60	0.11889033	$0.2 \cdot 10^{-7}$
13	0.90	3.10	$0.86372339 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
14	0.90	3.60	$0.65359456 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
15	0.90	4.60	$0.40964933 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
16	0.90	5.60	$0.27976357 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
17	1.20	1.20	0.41666664	$0.2 \cdot 10^{-7}$
18	1.20	2.00	0.22058820	$0.3 \cdot 10^{-7}$
19	1.20	2.80	0.12931032	$0.2 \cdot 10^{-7}$
20	1.20	3.60	$0.53333312 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
21	1.20	4.40	$0.57692288 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
22	1.20	5.20	$0.42134813 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
23	1.20	6.00	$0.32051264 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
24	1.20	6.80	$0.25167768 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
25	2.00	1.30	0.35149381	$0.3 \cdot 10^{-7}$
26	2.00	2.30	0.21528522	$0.3 \cdot 10^{-7}$
27	2.00	3.30	0.13431831	$0.2 \cdot 10^{-7}$
28	2.00	4.30	$0.88928392 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
29	2.00	5.30	$0.62324693 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
30	2.00	6.30	$0.45777047 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
31	2.00	7.30	$0.34910089 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
32	2.00	8.30	$0.27438588 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
33	5.00	1.40	0.18545992	$0.2 \cdot 10^{-7}$
34	5.00	2.50	0.15999998	$0.2 \cdot 10^{-7}$
35	5.00	3.60	0.13171757	$0.2 \cdot 10^{-7}$
36	5.00	4.70	0.10617963	$0.2 \cdot 10^{-7}$
37	5.00	5.80	$0.85266012 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
38	5.00	6.90	$0.68861020 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
39	5.00	8.00	$0.56179757 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
40	5.00	9.10	$0.46377870 \cdot 10^{-1}$	$0.2 \cdot 10^{-7}$

TABLE 10

$k$	$x_1^{(k)}$	$x_2^{(k)}$	$k$	$x_1^{(k)}$	$x_2^{(k)}$
1	$-0.11703344 \cdot 10^{-7}$	0.40000000	13	0.39999999	0.34641016
2	$0.35110033 \cdot 10^{-7}$	$-0.40000000$	14	$-0.39999997$	$-0.34641017$
3	0.80000000	0.00000000	15	0.69282032	0.20000000
4	$-0.80000000$	$-0.11703344 \cdot 10^{-7}$	16	$-0.69282031$	$-0.20000001$
5	$-0.56568540$	$-0.28284272$	17	0.20705522	0.38637033
6	0.56568542	0.28284271	18	$-0.20705520$	$-0.38637033$
7	0.56568545	$-0.28284269$	19	$-0.20705524$	0.38637032
8	$-0.56568543$	0.28284270	20	0.20705527	$-0.38637032$
9	0.69282034	$-0.19999998$	21	0.77274066	0.10352761
10	$-0.69282033$	0.19999999	22	$-0.77274065$	$-0.10352761$
11	$-0.40000001$	0.34641015	23	$-0.77274066$	0.10352760
12	0.40000003	$-0.34641015$	24	0.77274067	$-0.10352760$

/104

TABLE 11

$N_1$	$\xi_{N_1}$	$\eta_{N_1}$	$u(\xi_{N_1}, \eta_{N_1})$	$\epsilon$
1	0.10	0.85	-0.31638123	$0.4 \cdot 10^{-3}$
2	0.10	0.90	-0.26741077	$0.3 \cdot 10^{-3}$
3	0.10	0.95	-0.22799725	$0.2 \cdot 10^{-3}$
4	0.10	1.00	-0.19593167	$0.1 \cdot 10^{-3}$
5	0.10	1.05	-0.16958642	$0.9 \cdot 10^{-4}$
6	0.10	1.10	-0.14774571	$0.6 \cdot 10^{-4}$
7	0.40	1.20	-0.37494110	$0.2 \cdot 10^{-3}$
8	0.40	1.40	-0.24916701	$0.3 \cdot 10^{-4}$
9	0.40	1.60	-0.17299237	$0.2 \cdot 10^{-5}$
10	0.40	1.80	-0.12455676	$0.1 \cdot 10^{-4}$
11	0.40	2.00	-0.92448923 $\cdot 10^{-1}$	$0.7 \cdot 10^{-5}$
12	0.40	2.20	-0.70395625 $\cdot 10^{-1}$	$0.4 \cdot 10^{-4}$
13	0.50	0.85	-0.89872762	$0.2 \cdot 10^{-5}$
14	0.50	2.50	-0.59168693 $\cdot 10^{-1}$	$0.3 \cdot 10^{-4}$
15	0.50	3.00	-0.35060798 $\cdot 10^{-1}$	$0.1 \cdot 10^{-5}$
16	0.50	3.50	-0.22399381 $\cdot 10^{-1}$	$0.6 \cdot 10^{-6}$
17	0.50	4.00	-0.15147633 $\cdot 10^{-1}$	$0.3 \cdot 10^{-6}$
18	0.50	4.50	-0.10707785 $\cdot 10^{-1}$	$0.1 \cdot 10^{-6}$
19	0.70	0.80	-0.87710776	$0.2 \cdot 10^{-4}$
20	0.70	0.90	-0.74554028	$0.2 \cdot 10^{-4}$
21	0.70	1.50	-0.27969678	$0.2 \cdot 10^{-4}$
22	0.70	1.70	-0.20831214	$0.1 \cdot 10^{-4}$
23	0.70	1.90	-0.15822948	$0.1 \cdot 10^{-4}$
24	0.70	2.10	-0.12244221	$0.7 \cdot 10^{-5}$
25	1.00	0.80	-0.59488192	$0.2 \cdot 10^{-5}$
26	1.00	1.10	-0.45043029	$0.1 \cdot 10^{-4}$
27	1.00	1.30	-0.35929693	$0.1 \cdot 10^{-4}$
28	1.00	2.30	-0.11626231	$0.5 \cdot 10^{-5}$
29	1.00	2.40	-0.10503420	$0.4 \cdot 10^{-5}$
30	1.00	2.50	-0.86350308 $\cdot 10^{-1}$	$0.3 \cdot 10^{-5}$
31	1.50	0.80	-0.28736243	$0.1 \cdot 10^{-4}$
32	1.50	3.00	-0.71109321 $\cdot 10^{-1}$	$0.2 \cdot 10^{-5}$
33	1.50	3.70	-0.43685470 $\cdot 10^{-1}$	$0.9 \cdot 10^{-6}$
34	1.50	4.40	-0.28265522 $\cdot 10^{-1}$	$0.5 \cdot 10^{-6}$
35	1.50	5.10	-0.19157636 $\cdot 10^{-1}$	$0.2 \cdot 10^{-7}$
36	1.50	5.80	-0.13508267 $\cdot 10^{-1}$	$0.8 \cdot 10^{-7}$
37	2.00	0.80	-0.14863768	$-0.5 \cdot 10^{-5}$
38	2.00	0.90	-0.15560525	$-0.4 \cdot 10^{-5}$
39	2.00	1.00	-0.16000339	$-0.3 \cdot 10^{-5}$
40	2.00	2.00	-0.12499848	$0.2 \cdot 10^{-5}$
41	2.00	3.00	-0.71004589 $\cdot 10^{-1}$	$0.1 \cdot 10^{-5}$
42	2.00	4.00	-0.39999326 $\cdot 10^{-1}$	$0.7 \cdot 10^{-6}$
43	2.00	5.00	-0.23780922 $\cdot 10^{-1}$	$0.3 \cdot 10^{-6}$

INITIAL DATA FOR FIRST NUMERICAL EXAMPLE

/105

Address	Command					Comments
0000		30	0100		0010	Initial data input card.
1		31		0622	0007	
2		77			0060	
3			0622	0002	0622	
4		34			0134	

# INITIAL DATA

Address					Command	Comments
0622		0	+	2000		(b-b̄) is the difference bet. semiaxes S and S <sub>1</sub> .
2	+	0	+	8000		Semi-minor axis of basic ellipse S.
3	+				0005	N is the length of prog. for comput. boundary func.
4			0000	0050	0000	Values of normal derivative of unknown func. N <sub>1</sub> =
5						50(8)
6		03	0017	0017	0001	Beg. of prog. for computing boundary function.
7		03	0020	0020	0002	
0630		01	0001	0002	0001	
1		04	0017	0001	0001	
2		34			0555	End of program for computing boundary function.

# NUMERICAL INFORMATION

Address					Numbers	Address					Numbers
0633		+	0	+	8000	0670		+	1	+	4600
4		+	1	+	1000	1		+	0	+	9000
5		+	0	+	8000	2		+	1	+	5600
6		+	1	+	1200	3		+	1	+	1200
7		+	0	+	8000	4		+	1	+	1200
0640		+	1	+	1400	5		+	1	+	1200
1		+	0	+	8000	6		+	1	+	2000
2		+	1	+	1600	7		+	1	+	1200
3		+	0	+	8000	0700		+	1	+	2800
4		+	1	+	1800	1		+	1	+	1200
5		+	0	+	8000	2		+	1	+	3600
6		+	1	+	2000	3		+	1	+	1200
7		+	0	+	8000	4		+	1	+	4400
0650		+	1	+	2400	5		+	1	+	1200
1		+	0	+	8000	6		+	1	+	5200
2		+	1	+	2800	7		+	1	+	1200
3		+	0	+	9000	0710		+	1	+	6000
4		+	1	+	1100	1		+	1	+	1200
5		+	0	+	9000	2		+	1	+	6800
6		+	1	+	1600	3		+	1	+	2000
7		+	0	+	9000	4		+	1	+	1300
0660		+	1	+	2100	5		+	1	+	2000
1		+	0	+	9000	6		+	1	+	2300
2		+	1	+	2600	7		+	1	+	2000
3		+	0	+	9000	0720		+	1	+	3300
4		+	1	+	3100	1		+	1	+	2000
5		+	0	+	9000	2		+	1	+	4300
6		+	1	+	3600	3		+	1	+	2000
7		+	0	+	9000	4		+	1	+	5300
5		+	1	+	2000	0740		+	1	+	3600
6		+	1	+	6300	1		+	1	+	5000
7		+	1	+	2000	2		+	1	+	4700
0730		+	1	+	7300	3		+	1	+	5000
1		+	1	+	2000	4		+	1	+	5800
2		+	1	+	8300	5		+	1	+	5000
3		+	1	+	5000	6		+	1	+	6900
4		+	1	+	1400	7		+	1	+	5000
5		+	1	+	5000	0750		+	1	+	8000
6		+	1	+	2500	1		+	1	+	5000
7		+	1	+	5000	2		+	1	+	9100

# INITIAL DATA FOR SECOND NUMERICAL EXAMPLE

/106

Address	Command					Comments
0000		30	0100		0014	Initial data input card.
1		31		0622	0007	
2		77			0060	
3			0622	0002	0622	
4		34			0134	

## INITIAL DATA

Address	Command					Comments
0622	+	0	+	1000		(b-b) is difference bet. semiaxes S and S <sub>1</sub> .
3	+	0	+	5000		b is the semi-minor axis of basic ellipse S.
4					0011	N is length of prog. for computing boundary func.
5			0000	0053	0000	Values of norm. derivative of unknown func. N <sub>2</sub> =53(8)
6		03	0017	0017	0001	Beg. of program for computing boundary function.
7		03	0020	0020	0002	
0630		01	0001	0002	0001	
1		03	0001	0001	0001	
2		03	0017	0020	0002	
3		25	0002	0001	0002	
4		04	0002	0001	0001	
5		15	0001	0565	0001	
6		34			0555	End of prog. for computing boundary function.

## NUMERICAL DATA

Address	Numbers				Address	Numbers			
0637	+	0	+	1000	6	+	1	+	1000
0640	+	0	+	8500	7	+	0	+	1000
1	+	0	+	1000	0650	+	1	+	1050
2	+	0	+	9000	1	+	0	+	1000
3	+	0	+	1000	2	+	1	+	1100
4	+	0	+	9500	3	+	0	+	4000
5	+	0	+	1000	4	+	1	+	1200
5	+	0	+	4000	2	+	1	+	1100
6	+	1	+	1400	3	+	1	+	1000
7	+	0	+	4000	4	+	1	+	1300
0660	+	1	+	1600	5	+	1	+	1000
1	+	0	+	4000	6	+	1	+	2300
2	+	1	+	1800	7	+	1	+	1000
3	+	0	+	4000	0730	+	1	+	2400
4	+	1	+	2000	1	+	1	+	1000
5	+	0	+	4000	2	+	1	+	2500
6	+	1	+	2200	3	+	1	+	1500
7	+	0	+	5000	4	+	0	+	8000

NUMERICAL DATA (con't)

Address	Numbers				Address	Numbers			
0670	+	0	+	8500	5	+	1	+	1500
1	+	0	+	5000	6	+	1	+	3000
2	+	1	+	2500	7	+	1	+	1500
3	+	0	+	5000	0740	+	1	+	3700
4	+	1	+	3000	1	+	1	+	1500
5	+	0	+	5000	2	+	1	+	4400
6	+	1	+	3500	3	+	1	+	1500
7	+	0	+	5000	4	+	1	+	5100
0700	+	1	+	4000	5	+	1	+	1500
1	+	0	+	5000	6	+	1	+	5800
2	+	1	+	4500	7	+	1	+	2000
3	+	0	+	7000	0750	+	0	+	8000
4	+	0	+	8000	1	+	1	+	2000
5	+	0	+	7000	2	+	0	+	9000
6	+	0	+	9000	3	+	1	+	2000
7	+	0	+	7000	4	+	1	+	1000
0710	+	1	+	1500	5	+	1	+	2000
1	+	0	+	7000	6	+	1	+	2000
2	+	1	+	1700	7	+	1	+	2000
3	+	0	+	7000	0760	+	1	+	3000
4	+	1	+	1900	1	+	1	+	2000
5	+	0	+	7000	2	+	1	+	4000
6	+	1	+	2100	3	+	1	+	2000
7	+	1	+	1000	4	+	1	+	5000
0720	+	0	+	8000					
1	+	1	+	1000					

/107

OPERATIONAL CODE FOR METHOD OF FUNCTIONAL EQUATIONS

Address	Command					Address	Command				
0000		00				0020		71			
1		30	0100		1647	1		35	0050	0041	0041
2		31		0014	0050	2		33		0002	0024
3		35	0050	0005	0006	3		34			0015
4		33	0001			4		30	0100		0004
5		42	0337	2154	1211	5		31		0001	
6		30	1400		1043	6		34			0001
7		31		0342	0050	7		30	0400		1043
0010		30	0400		1043	0030		31		0060	0007
1		71			0044	1		35	0007	0044	0474
2		35	0050	0044	0015	2		33		0003	
3		33		0001		3		34			0027
4		34			0006	4		30	0400	1044	0255
5		30	1400	1044	0255	5		31		0340	0007
6		31		1406	0050	6		35	0007	0041	0340
7		30	0400	1044	0255	7		33		0004	

OPERATIONAL CODE FOR METHOD OF FUNCTIONAL EQUATIONS

(con'td)

/108

Address	Command				Address	Command			
0040	34			0034	0140	62	0624		0042
1	00				1	22	0042	0124	0344
2	00				2	22	0316	0344	0144
3	00				3	22	0317	0344	0146
4	00				4	00			
5	00				5	31		0626	0046
6	00				6	00			
7	00				7	71			0010
0050	00				0150	35	0046	0010	0153
1	00				1	33		0005	
2	00				2	34			0144
3	00				3		0146		0054
4	00				4	22	0314	0042	0344
5	00				5	62	0322	0344	0344
6	00				6	17	0344	0124	0345
7	00				7	35	0345		0161
0060	34		0002	0061	0160	62	0344	0124	0344
1	26	0002	0026	0063	1		0344		0055
2	22	0124	0063	0063	2	26	0344	0013	0345
3	33		0002	0001	3	22	0345	0315	0057
4	17	0125	0001	0004	4	17	0625	0320	0047
5	20	0001	0004	0076	5	17	0625	0125	0346
6	22	0124	0076	0076	6	75	0346		0232
7	22	0125	0004	0001	7	17	0625	0321	0345
0070	22	0126	0001	0122	0170	22	0346	0346	0346
1	22	0002	0124	0063	1	26	0346	0113	0346
2	76	0001	0133	0117	2	36	0346	0344	0225
3	33		0002		3	62	0346	0344	0346
4	17	0126	0076	0001	4		0124		0343
5	35		0001	0063	5	62	0344	0124	0347
6	33		0002	0002	6	22	0323	0347	0177
7	22	0076	0126	0076	7	00			
0100	22	0124	0122	0122	0200	31		0623	0010
1	45		0001	0002	1	22	0324	0345	0203
2	26	0002	0106	0004	2	77			0060
3	03	0127	0002	0002	3	00			
4	17	0130	0001	0003	4	00			
5	76	0131	0003	0073	5	22	0325	0347	0210
6	41	0132	0004	0004	6	22	0327	0344	0212
7	01	0003	0002	0002	7	22	0330	0344	0213
0110	66	0001	0004	0001	0210	00			
1	76		0001	0103	1	31		0623	0010
2	25	0002	0004	0001	2	00			
3	36	0132	0004	0121	3	00			
4	04	0001	0127	0001	4	31		0623	0010
5	42	0004	0132	0004	5	35	0010	0326	0220
6	35			0113	6	33		0006	
7	03	0001	0127	0001	7	34			0210
0120	41	0004	0132	0004	0220	35	0342	0343	0027
1	36	0125	0004	0072	1	22	0330	0057	0330
2	33		0002	0003	2	22	0045	0124	0045
3	35			0074	3	00			
4				0001	4	34			0172
5				3777	5				0343
6	37		3777		6		0346		0344
7	04	1200			7	26	0346	0013	0345
0130		3700			0230		0344		0053
1		1100			1	34			0175
2		2040			2		0331		0344
3	34	1463	0631	2315	3	26	0331	0013	0345
4				0045	4	34		0204	0175
5				0343	5				0344
6	14		0342	0043	6	22	0211	0337	0211
7		0622		0051	7	22	0327	0331	0327
		0623							

OPERATIONAL CODE FOR METHOD OF FUNCTIONAL EQUATIONS (con't)

/109

Address	Command					Address	Command				
0240			0211		0214	0340			0003		0626
1			0625		0001	1			0005		0627
2	25		0001	0077	0345	2	04	0001	0126		0004
3	02		0332	0345	0332	3	14	0004	0003		0004
4			0623		0002	4	41	0004	0130		0004
5			0002		0003	5	26	0004	0046		0002
6			0334		0304	6	35	0002			0066
7			0624		0004	7	02	0127	0003		0003
0250			0004		0005	0350	26	0004	0036		0005
1	03		0005	0005	0006	1	03	0003	0121		0004
2	03		0051	0051	0007	2	03	0004	0004		0003
3	04		0006	0007	0006	3	01	0003	0122		0002
4	03		0003	0003	0007	4	03	0002	0002		0002
5	01		0006	0007	0006	5	01	0002	0003		0003
6	36		0006	0332	0276	6	01	0003	0123		0003
7	00					7	01	0002	0124		0002
0260	01		0005	0001	0005	0360	03	0002	0003		0002
1	34			0257	0251	1	02	0125	0002		0003
2			0335		0304	2	04	0004	0003		0002
3	02		0004	0001	0004	3	25	0002	0077		0003
4	34			0257	0250	4	03	0003	0003		0004
5						5	01	0127	0004		0003
6	01		0003	0001	0003	6	04	0002	0003		0002
7	34			0257	0246	7	02	0127	0004		0004
0270			0336		0265	0370	04	0004	0003		0003
1	02		0002	0001	0002	1	22	0117	0005		0004
2	34			0257	0245	2	76	0004	0120		0114
3					0343	3		0002			0004
4					0053	4		0003			0002
5			0344		0306	5		0004			0003
6			0003		0626	6	15	0002	0005		0002
7			0005		0627	7	22	0005	0127		0004
0300			0276	0333	0276	0400	15	0003	0004		0003
1	22		0277	0333	0277	1	37	0400			
2	22		0344	0333	0344	2	37	0600			
3	35		0344	0055	0305	3	75	1552	3107		0735
4	00					4	75	3117	0500		1322
5			0124		0343	5	74	3210	3556		1027
6	62		0344	0124	0347	6		1322	0647		1264
7	34			0223	0205	7	75	1621	2273		2707
0310					0344	0410		1444	0773		1242
1			0340		0276	1	01	1000			
2			0341		0277	2		1000			
3					0304	3	36	0001	0161		0153
4	34				0626	4	06	0001	0001		0002
5						5	01	0002	0162		0003
6			0001			6	02	0002	0162		0004
7	30		1400	1330		7	04	0004	0003		0003
0320	30		0400	1330		0420	43	0003	0003		0002
1			3777			1	43	0155	0002		0004
2						2	41	0004	0156		0004
3					3406	3	43	0004	0002		0004
4	30		0100			4	41	0004	0157		0004
5			0623		0623	5	43	0004	0002		0004
6	30		1403			6	01	0004	0160		0004
7	77		3777	3777	3777	7	03	0004	0003		0002
0330	20		0010	0326	0623	0430	07	0001			0004
1	30		1401	0000		1	02	0004	0130		0004
2						2	03	0004	0163		0004
3	01		1000			3	01	0004	0002		0002
4						4	35				0154
5					0003	5	33		0013		
6	34				0260	6	37				
7	34				0263	7	12		1150		0200
	34			0003	0271						

OPERATIONAL CODE FOR METHOD OF FUNCTIONAL EQUATIONS (con't)

/110

Address	Command					Address	Command				
0440	06	0006	1450	0074		0540	03	0334	0333	0223	
1	02	0252	2526	3155		1	01	0331	0333	0227	
2	01	1777	3777	3775		2		0227		0001	
3	41	1000				3	00				
4		1324	0236	1464		4	01	0335	0001	0242	
5		1305	3102	3770		5		0001		0335	
6	14		0004	0002		6	06	0333	0262	0337	
7	15	0001	0001	0003		7	03	0242	0337	0220	
0450	76	0127	0003	0172		0550	06	0333	0262	0337	
1	03	0003	0207	0003		1	01	0331	0337	0247	
2	76	0003	0127	0207		2		0216		0251	
3	65		0067	0004		3		0247		0001	
4	65	0004	0001	0004		4	00				
5	76	0003	0210	0200		5	01	0251	0001	0251	
6	04	0003	0211	0003		6	01	0247	0333	0247	
7	35			0172		7	36	0247	0227	0271	
0460	26	0002	0004	0002		0560	05	0251	0274	0251	
1	03	0003	0210	0003		1	01	0242	0251	0242	
2	14	0003	0003	0005		2	01	0242	0251	0251	
3	22	0002	0005	0002		3	04	0333	0330	0337	
4	76	0002	0212	0176		4	03	0251	0337	0251	
5	47	0004		0004		5	06	0333	0300	0333	
6	26	0004	0006	0004		6	02	0251	0220	0337	
7	55	0004	0001	0004		7	76	0251	0127	0307	
0470	22	0004	0002	0002		0570	04	0337	0251	0337	
1	37	0734	3262	2000		1		0251		0220	
2	04	1200				2	76	0223	0337	0266	
3	04	1177	3777	3777		3	00				
4		0100				4	00				
5	34		0336	0214		5		0227		0331	
6	26	0336	0026	0335		6	05	0333	0302	0333	
7	14	0322	0223	0312		7	34			0252	
0500	40	0323		0227		0600		0216		0337	
1	16	0335	0316	0220		1					
2	00					2	01		0220		
3	17	0337	0326	0242		3				0247	
4	16	0242	0223	0223		4				0331	
5	00					5	74		0227	0221	
6	62	0223	0242	0223		6				0001	
7	22	0223	0324	0223		7		0001			
0510	26	0337	0013	0337		0610		3777			
1	00					1				0003	
2	34		0227	0221		2	03	1400			
3	22	0335	0325	0335		3	00				
4	34		0227	0217		4	00				
5	17	0337	0326	0337		5	00				
6	26	0337	0126	0311		6	00				
7	16	0317	0311	0251		7	00				
0520	16	0337	0311	0311		0620	00				
1	16	0320	0311	0311		1	00				
2	22	0335	0325	0335		2	25		0040	0002	
3	16	0335	0321	0242		3	76	0001	0343	0366	
4	00					4	06	0001	0001	0005	
5		0247		0261		5	41	0005	0370	0003	
6		0247		0272		6	36	0371	0005	0346	
7	22	0336	0327	0336		7	42	0003	0372	0003	
0530		0331		0001		0630	17	0001	0373	0005	
1	00					1		0371		0004	
2		0001		0335		2	36	0005	0127	0352	
3	00					3		0367		0004	
4	02	0332	0331	0337		4	03	0004	0003	0003	
5	36	0333	0337	0256		5	04	0005	0003	0004	
6		0337		0333		6	01	0003	0004	0003	
7		0336		0312		7	04	0005	0003	0005	

OPERATIONAL CODE FOR METHOD OF FUNCTIONAL EQUATIONS (con't)

/111

Address	Command					Address	Command				
0640	65	0003	0076	0003		0740	03	1311		2442	
1	01	0003	0005	0005		1			3324		
2	17	0001	0002	0003		2	15	2456	0461	2460	
3	26	0001	0101	0002		3	01	3000			
4	16	0003	0002	0002		4	03	2457		2442	
5	05	0005	0002	0002		5		3323			
6	36	0343	0001	0366		6		1311			
7	33		0010			7	03		2442	2442	
0650	37					0750	03	2510	2511	2442	
1		1473	2600			1		0002			
2		1410				2	04	0127	0002		
3		1111	1040			3	03		2511		
4	77	0034	1400			4				3777	
5	01	1777	3777	3777		5	30	0410		0372	
6				0030		6	02	0051	0043	0052	
7				2453		7	04	0052	0051	0050	
0660				2454		0760				2445	
1		3242				1	04	0406	0405	0001	
2		3243				2	77			0060	
3		0031				3	03	0003	0050	3244	
4	40					4	03	0002	0052	3245	
5	03	1444	0773	1242		5	72			2446	
6	72	1217	1341	1075		6		3244		0001	
7	54	1524	2377	1607		7				0164	
0670	00					0770	32	0002			
1	06	1624	2734	0260		1	22	0504	0325	0504	
2	07	1320				2	22	2446	0324	2446	
3	11	1034				3	75	2446	0436	0504	
4	10					4	22	0477	0325	0477	
5	10	1320				5	22	0501	0436	0501	
6	06	1320				6	22	0502	0436	0502	
7	11	1166				7	22	2445	0324	2445	
0700	10	1034				1000	75	2445	0374	0477	
1	11	1224				1	72	0467		2445	
2	10	1130				2		0325		2446	
3	07	1700				3		0467		2447	
4	11	1130				4	22	2445	0467	2445	
5	06	1700				5	77			0213	
6	10	1700				6		0400	0401	0402	
7	05	1700				7		0403	0404		
0710	10	1510				1010	34		0567	0540	
1	07	1130				1		0404		1312	
2	10	1774				2	22	0527	0324	0527	
3	07	1510				3	22	2447	0467	2447	
4	11	1072				4	75	2447	2445	0523	
5	04	1700				5	22	0541	0467	0541	
6	10	1414				6	22	0542	0467	0542	
7	10	1224				7	22	2446	0325	2446	
0720	11	1262				1020	75	2446	0377	0521	
1				0002		1	34			0570	
2	77	3777	3777	3777		2				0556	
3			0001			3		3244		2453	
4	03	0002	3324	2462		4		3245		2454	
5	73	1467	2145	2625		5	77			0060	
6	03	1441	0131	0455		6	02	0003	2453	2455	
7			0373			7	03	2455	2455	2455	
0730		2455		2500		1030	03	0002	0051	2457	
1		3244		2460		1	02	2457	2454	2457	
2		3245		2461		2	03	2457	2457	2457	
3		0001	0001			3	01	2455	2457	2455	
4	03			0372		4		0001		2452	
5	63	2500	1312	2456		5		2455		0001	
6		1506	3342	3530		6	77			0131	
7	03	2456		2511		7	25	0002	0077	2455	
		2510	2510	2442							

OPERATIONAL CODE FOR METHOD OF FUNCTIONAL EQUATIONS (con't)

/112

Address	Command				Address	Command			
1040	00	2455		2456	1140	01	2456	2442	2456
1	22	0375	2447	0561	1	22	2457	0325	2457
2	00				2	75	2457	2452	0667
3	22	0376	2447	0563	3	00			
4	00				4	22	0661	0324	0661
5	22	2452		0001	5	22	2452	0325	2452
6	34		0556	0543	6	62	2451	0440	2451
7	03	2455	2456	0001	7	75	2452	2443	0647
1050	00				1150	34			0673
1		1312		0001	1	62	2454	0325	2454
2	77			0340	2	62	2453	2455	2453
3	04	0127	0002	3324	3	62	2455	0440	2455
4	22	0325	0325	2443	4	34			0654
5		0454		0661	5	22	2450	2443	2450
6		0455		0703	6	22	2450	0464	2446
7	62	2443	0325	2444	7		2444		2451
1060	62	2444	0325	2445	1160	2443			2452
1		0325		2446	1	0465			2453
2		0325		2447	2				2454
3				2450	3	15			2455
4	22	2450	0325	2450	4	16	2446	2453	0704
5	22	2447	2450	2447	5	00			
6	75	2450	2444	0603	6	00			
7		0456		2450	7	01	2455	2442	2455
1070		0457		2451	1170	22	2454	0325	2454
1		0460		0623	1	35	2454	2443	0714
2		0325		2452	2	22	0703	0450	0703
3		0440		2453	3	62	2446	2452	2446
4	22	2450	2447	2454	4	62	2452	0325	2452
5				2455	5	34			0702
6	15			2456	6	22	2447	2445	2447
7	16	2451	2454	0617	7	22	2447	0464	2446
1100	00				1200	0466			2454
1	01	2456	2442	2456	1	15			2452
2	22	2455	0325	2455	2				2456
3	75	2455	2446	0631	3		2454		0723
4	00				4	16	2446	2453	0725
5	22	0623	0324	0623	5	00			
6	22	2446	0325	2446	6	25	2442	0001	2442
7	22	2451	0440	2451	7	00			
1110	75	2446	2543	0613	1210	01	2452	2442	2452
1	34			0634	1	22	2456	0325	2456
2	22	2454	2452	2454	2	35	2456	2451	0734
3	22	2451	2453	2451	3	22	0723	0440	0723
4	34			0616	4	62	2446	0325	2446
5		0325		2446	5	34			0722
6		0127		2510	6	22	2454	0450	2454
7				2450	7	62	2446	0467	2446
1120	22	2450	2446	2450	1220	62	2451	0325	2451
1	35	2446	2444	0643	1	75	2451		0720
2	22	2446	0325	2446	2	22	2450	0463	2450
3	34			0637	3	26	2450	0126	2453
4	22	0462	2444	2446	4		0470		2451
5	22	2450	0463	2451	5	16	2453	2451	0746
6	26	2451	0113	2451	6	01	2455	2452	0001
7		0325		2452	7	77			0340
1130		2451		2453	1230	00			
1		2446		2454	1		0325		2446
2	26	2445	0113	2455	2		0471		2452
3				2457	3	22	0463	2447	2451
4				2456	4	26	2451	0126	2451
5	15			0655	5	16	2452	2451	0755
6	16	2453	2454		6	16	0755	2450	0755
7	00				7	00			

OPERATIONAL CODE FOR METHOD OF FUNCTIONAL EQUATIONS (con't)

/113

Address	Command					Address	Command				
1240	22	2446	0325	2446		1334	33		0010		
1	35	2446	2443	0762		1	34			1053	
2	22	0755	0472	0755		2		0452		1070	
3	34			0755		3	00				
4	22	2453	0325	2443		4	31		2500	0010	
5	75	2443	0377	0574		5	35	0010	0437	1066	
6		0324		2463		6	33		0011		
7		0441		2464		7	34			1061	
1250	14		2446	2465		1350				2454	
1		0445		1022		1	15			2452	
2	15			2454		2	00				
3		2464		1012		3	01	2456	2452	2452	
4		0446		1002		4	22	1070	0450	1070	
5		0447		1003		5	22	2454	0324	2454	
6				2445		6	75	2454	0451	1070	
7	15			2455		7	03	2452	0442	2452	
1260		2454		0001		1360	22	2455	0324	2455	
1	77			0060		1	35	2455	2453	1106	
2		0003		2456		2	76	2452	0453	1122	
3	03	0002	0051	2457		3	72	2452		0001	
4	00					4	77			0164	
5	00					5	32	0002			
6	02	2456	2460	2460		6	33			0001	
7	03	2460	2460	2460		7	34			1122	
1270	02	2457	2461	2461		1370	02	0127	2452	2452	
1	03	2461	2461	2461		1	76	2452	0453	1115	
2	01	2460	2461	0001		2	72	2452		0001	
3	77			0131		3	77			0164	
4	00					4	32	0002			
5	01	2455	2462	2455		5	32	0324			
6	22	1002	0467	1002		6	33			0002	
7	22	1003	0467	1003		7	22	1053	0444	1053	
1300	22	1012	0440	1012		1400	22	2453	0324	2453	
1	22	2445	0324	2445		1	22	2450	0324	2450	
2	75	2445	2463	1002		2	75	2450	0374	1051	
3	25	2455	0077	2455		3	34			0034	
4	00					4	22	1061	0444	1061	
5	22	1022	0324	1022		5	34			1060	
6	01	2454	0442	2454		6		0054		0341	
7	36	2454	0443	0771		7	00				
1310	30	1403		0371		1410	31		0626	0010	
1	31		2500	0010		1	35	0010	0046	0346	
2	20	0010	0437	3072		2	33		0012		
3	30	1410		0372		3	34			0341	
4	31		2500	0010		4				0341	
5	30	0410		0372		5		3244		0014	
6	71			0010		6		3245		0015	
7	35	0010	0437	1040		7	77			0213	
1320	33		0007			1420		0557	0560	0561	
1	34			1026		1		0562	0563		
2	22	1031	0444	1031		2	34		0556	0535	
3	22	1033	0444	1033		3				3214	
4	22	2463	0324	2463		4	22	0563	0615	0347	
5	22	2465	0440	2465		5	22	0347	0615	0350	
6	22	2464	2465	2464		6	22	0350	0615	0355	
7	22	2446	0324	2446		7	22	0355	0324	0355	
1330	75	2446	0374	0767		1430	22	0341	0324	0341	
1				2450		1	75	0341	0564	0347	
2		0324		2453		2		0324		0216	
3				2455		3		0566		0217	
4		0473		1061		4		0567		0220	
5	30	0410		0372		5	15			0222	
3	31		1312	0010		6	16	0217	0220	0221	
7	35	0010	0437	1060		7				0371	

OPERATIONAL CODE FOR METHOD OF FUNCTIONAL EQUATIONS (con't)

/114

Address	Command					Address	Command				
1440		01	0221	0215	0221	1540					
1		22	0217	0325	0217	1	72	0607		0503	
2		22	0220	0570	0220	2		0610		0513	
3		22	0222	0324	0222	3	15			0220	
4		75	0222	0216	0370	4	15			0221	
5			0221		3164	5	34		0556	0534	
6		22	0377	0324	0377	6		0001		0010	
7		22	0216	0324	0216	7		0011		0001	
1450		75	0216	0571	0365	1550	77			0131	
1					3406	1					
2		22	0403	0324	0403	2	25	0001	0077	0001	
3		75	0403	0570	0403	3	02	0001	0010	0001	
4			0572		0216	4	01	0221	0001	0221	
5					0217	5	22	0503	0570	0503	
6			0573		0220	6	01	0220	0611	0220	
7			0574		0422	7	36	0220	0612	0477	
1460			0221		0221	1560	04	0221	0613	0016	
1		30	0410		0372	1					
2		31		2216	0010	2	77			0164	
3		35	0010	0565	0420	3	32	0002			
4		33		0013		4	22	0513	0325	0513	
5		34			0413	5	75	0513	0614	0513	
6		16	0220	0216	0421	6	22	0471	0615	0471	
7						7	22	0472	0615	0472	
1470						1570	22	0602	0603	0602	
1		22	0422	0575	0422	1	75	0602	0216	0471	
2		22	0220	0325	0220	2					
3		22	0221	0324	0221	3					
4		75	0221	0576	0420	4	22	0606	0057	0606	
5		22	0413	0577	0413	5	22	0217	0324	0217	
6		22	0216	0570	0216	6	75	0217	0045	0460	
7		22	0217	0324	0217	7	35	0053		0533	
1500		75	0217	0564	0410	1600		0053		0216	
1		35	0047		0452	1	34		0524	0460	
2		35	0047	0325	0437	2	33	0002			
3		35	0047	0600	0445	3		0220		0001	
4			0601		0444	4				0060	
5			3164		0001	5	03	0003		0017	
6		77			0164	6	03	0051	0002	0020	
7		32	0002			7		0051	0003	0010	
1510		22	0437	0325	0437	1610		0002		0011	
1		75	0437	0602	0437	1	03	0010	0014	0010	
2		34			0452	2	03	0011	0015	0011	
3			3406		0001	3	01	0010	0011	0010	
4		77			0164	4	02	0051	0010	0010	
5		32	0002			5	02	0017	0014	0011	
6		22	0445	0325	0445	6	03	0011	0011	0011	
7		75	0445	0324	0445	7	02	0020	0015	0012	
1520		26	0042	0013	0215	1620	03	0012	0012	0012	
1		22	0342	0215	0462	1	01	0012	0011	0011	
2		26	0215	0013	0215	2	04	0010	0011	0010	
3		35	0045		0531	3	34			0626	
4			0055		0216	4	03	0010	0001	0001	
5					0217	5	00				
6		22	0606	0216	0461	6	40				
7						7	03	1444	0773	1242	
1530						1630	72	1217	1341	1075	
1		35	0010	0565	0466	1	54	1524	2377	1607	
2		33		0014		2	00				
3		34			0461	3				0030	
4		22	0604	0215	0471	4	77	3777	3777	3777	
5		22	0605	0215	0472	5	03	3324		0215	
6					0602	6			3214		
7						7			0001		
										0031	

Address	Command					Address	Command				
1640				3164		3		0627			0015
1	03	2216		0222		4	30	0401			
2	01	0222	3406	3406		5	03	0002	3406	0001	
3			0001	0001		6		0014		0001	
4				0372		7	73	1467	2145	2413	
5			0373		1660	03	1441	0131		0362	
6		0010			1	10	1750				
7	72				2		0017			0001	
1650		3214		0001	3		0002				
1				0002							
2		0626		0014							

§ 12. SOLUTION TO THE TWO-DIMENSIONAL EXTERNAL DIRICHLET  
PROBLEM BY THE METHOD OF GENERALIZED FOURIER SERIES

Computer Program

The computational process consists of the following steps:

- (1) Computation of uniformly distributed points on auxiliary ellipse;
- (2) Computation of scalar products;
- (3) Computation of coefficients of orthonormalization;
- (4) Construction of orthonormalized systems;
- (5) Checking accuracy of this orthonormalization;
- (6) Computation of Fourier coefficients  $B_i$ ;
- (7) Computation of coefficients  $C_i$ .
- (8) Finding harmonic function outside ellipse  $s$ .

Let us take two confocal ellipses:

the basic ellipse  $s$ :  $s: x_1 = a \cos \varphi$ ,  $x_2 = b \sin \varphi$  and the auxiliary ellipse

$$s_1: x_1^{(i)} = \bar{a} \cos \alpha_i, \quad x_2^{(i)} = \bar{b} \sin \alpha_i \quad (1);$$

$$M(x_1^{(i)}, x_2^{(i)}) \in s_1, \quad i = 1, 2, \dots, 24.$$

The points  $M(x_1^{(i)}, x_2^{(i)})$  are distributed on the auxiliary ellipse uniformly with an interval of  $\pi/12$ ; the values of the arguments  $\alpha_i$  ( $i=1, 2, \dots, 24$ ) are given in Table 7.

The points  $M(x_1^{(i)}, x_2^{(i)})$  are computed and printed out in the following sequence  $x_1^{(1)}, x_2^{(1)}; x_1^{(2)}, x_2^{(2)}; \dots; x_1^{(24)}, x_2^{(24)}$ . The scalar products

$$\begin{aligned} (\varphi_i(y), \varphi_k(y)) &= \int_{S_1} \varphi_i \varphi_k ds_1, \\ \varphi_i &= \frac{1}{2} \ln \left[ (x_1 - x_1^{(i)})^2 + (x_2 - x_2^{(i)})^2 \right] \quad i=1, 2, \dots, 24; \quad k=1, 2, \dots, i, \end{aligned} \quad (2)$$

are computed, as are the coefficients of orthonormalization  $A_{ik}$  by the following formulas:

$$A_{k,k} = \frac{1}{\|g_k\|}, \quad k=1, 2, \dots, 24 \quad (3_1)$$

$$A_{k,k-1} = A_k \sum_{j=1}^i \alpha_{k,k-j} A_{k-j,k-i}, \quad \begin{aligned} k &= 2, 3, 4, \dots, 24 \\ i &= 1, 2, 3, \dots, k-1 \end{aligned} \quad (3_2)$$

where

$$\begin{aligned} \|g_k\| &= \sqrt{\int_S \left[ \varphi_k + \sum_{j=1}^{k-1} B_{k,j} \varphi_j \right]^2 ds_y} = \\ &= \sqrt{\sum_{j=0}^{k-1} B_{k,k-j}^2 \int_S \varphi_{k-j}^2 ds_y + 2 \sum_{j=0}^{k-2} B_{k,k-j} \sum_{i=j+1}^{k-1} B_{k,i} \int_S \varphi_{k-j} \varphi_i ds_y}, \quad (3_3) \quad \underline{/116} \\ B_{k,k} &= 1, \quad k=2, 3, 4, \dots, 24 \end{aligned}$$

$$\alpha_{k, k-j} = - \sum_{i=1}^{k-j} A_{k-j, i} (\varphi_k(y) \varphi_i(y)), \quad \begin{matrix} k=2, 3, 4, \dots, 24 \\ j=k-1, k-2, \dots, 1 \end{matrix} \quad (3_4)$$

$$B_{k, j} = \sum_{i=1}^{k-j} \alpha_{k, k-i} A_{k-i, i}, \quad \begin{matrix} k=2, 3, 4, \dots, 24 \\ j=k-1, k-2, \dots, 1 \end{matrix} \quad (3_5)$$

After this we construct the orthonormalized system

$$\psi_i(\alpha_j) = \sum_{k=1}^i A_{i, k} \varphi_k(\alpha_j) \quad \begin{matrix} \alpha_j = (j-1) \frac{2\pi}{250} \\ i=1, 2, \dots, 24 \\ j=1, 2, \dots, 250 \end{matrix} \quad (4)$$

and the orthonormalization is checked within an accuracy of  $10^{-4}$

$$\begin{aligned} 1 - \int_0^{2\pi} \psi_i \psi_k ds &= 1 - \frac{2\pi}{250} \sum_{j=1}^{250} \psi_i(\alpha_j) \psi_k(\alpha_j) < 10^{-4} \quad \text{when } i = k, \\ \int_0^{2\pi} \psi_i \psi_k ds &= \frac{2\pi}{250} \sum_{j=1}^{250} \psi_i(\alpha_j) \psi_k(\alpha_j) < 10^{-4} \quad \text{when } i \neq k. \end{aligned} \quad (5)$$

Then we compute the Fourier coefficients

$$B_i = \int_{s_1} f(\alpha) \psi_i ds_1 = \frac{2\pi}{250} \sum_{j=1}^{250} f(s_j) \psi_i(\alpha_j), \quad i=1, 2, \dots, 24, \quad (6)$$

$$\alpha_j = (j-1) \frac{2\pi}{250},$$

where  $f(\alpha)$  is the boundary condition and  $\psi_{i,j}$  are the orthonormalized functions.

After this we compute the coefficients

$$C_i = \sum_{k=i}^{24} B_k A_{i, k}, \quad i=1, 2, \dots, 24, \quad (7)$$

where  $A_{i, k}$  are the coefficients of orthonormalization.

Outside the ellipse we seek the harmonic function  $u(\xi, \eta)$  in the following manner:

$$u(M) = \sum_{i=1}^{24} C_i \varphi_i(M), \quad (8)$$

where

$$\varphi_i(M) = \frac{1}{2} \ln \left[ (x_1^{(i)} - \xi)^2 + (x_2^{(i)} - \eta)^2 \right], \quad M(\xi, \eta) \in G_e.$$

The program consists of the following basic parts:

- (1) Control program;
- (2) A program for processing initial and numerical data;
- (3) First part of fundamental program;
- (4) Second part of fundamental program;
- (5) Finding the harmonic function outside ellipse s.

/117

The program operates in the following sequence: the entire program is fed into the machine. If the read-in is correct, the control program records on the zero magnetic drum the first part of the program in cells 0000 - - 1114, and the second part of the program in cells 1115 - 1331, automatically reads in the initial data input card, which in turn reads in

the initial data in binary notation, converts  $b$  and  $b - \bar{b}$  from the decimal notation system into the binary, and the control transmits the program for processing the numerical data by the command 0011.

The program for interpreting the initial and numerical data transfers  $b$  to cell 0051, and  $b - \bar{b}$  to cell 0043, and the program for computing the boundary function  $f(\alpha)$  is transferred from cell 0626 - 0626 +  $(n - 1)$  (where  $n$  is the length of the program in the octal notation system) to the zero magnetic drum is cells 1332 - 1332 +  $(n - 1)$ . (In compiling the program for  $f(\alpha)$ , the usable working cells must certainly be allowed for in the length of the program.) After this the program for processing the initial and numerical data of the coordinates of these points is converted from the decimal notation system to the binary and transferred to the zero magnetic drum from the cell 2462 -  $[1420 - n \cdot 2^{-1}]_8$  (if such a number of points exists).

After completion of the operation, the program for processing the numerical data transmits control to the control program by the command 0225, after which the first part of the basic program is read out from the drum.

The first part of the program computes the semiaxes of the auxiliary ellipse  $\bar{b} = b - (b - \bar{b})$  and  $a = \bar{b}/b$ , and computes and prints out the auxiliary points  $x_1^{(1)}, x_2^{(1)}; x_1^{(2)}, x_2^{(2)}; \dots; x_1^{(24)}, x_2^{(24)}$ . The scalar products  $\int_{s_1} \varphi_i \varphi_k ds_1$ ,  $i = 1, \dots, 24; k = 1, 2, \dots, i$  are computed from the standard program. We must mention that only the left-hand side of the matrix is computed.

$$\begin{aligned} & \int_{s_1} \varphi_1 \varphi_1 ds_1, \\ & \int_{s_1} \varphi_2 \varphi_1 ds_1, \int_{s_1} \varphi_2 \varphi_2 ds_1, \\ & \dots \dots \dots \\ & \int_{s_1} \varphi_{24} \varphi_1 ds_1, \int_{s_1} \varphi_{24} \varphi_2 ds_1, \dots, \int_{s_1} \varphi_{24} \varphi_{24} ds_1. \end{aligned}$$

Then the coefficients of orthonormalization  $A_{i,k}$  are computed from Formulas (3<sub>1</sub>) and (3<sub>2</sub>). In computing the coefficients of orthonormalization, there may be a stop from the command 0343 — if the machine is in good working order, this means that a certain coefficient  $A_{k,k}$  or  $A_{k,k-1}$  is found to be negative, and in such case it is recommended to change the dimensions of the auxiliary boundary  $s_1$ . /118

Then the orthonormalized system is computed from Formula (4) and the orthonormalization is verified within an accuracy of  $10^{-4}$  from Formula (5). If the orthonormalization is not carried out within the required accuracy, then the following is printed out

$$1 - \frac{2\pi}{250} \sum_{j=1}^{250} \varphi_i^2(\alpha_j) > 10^{-4}, \quad i=1, 2, \dots, 24, \quad \alpha_j = (j-1) \frac{2\pi}{250}$$

and the conditional unit, i.e., the diagonal term, does not reach the required accuracy or

$$\frac{2\pi}{250} \sum_{j=1}^{250} \varphi_i(\alpha_j) \varphi_k(\alpha_j) > 10^{-4}$$

is printed out when  $i \neq k$ , i. e. the nondiagonal term does not reach the required accuracy.

If the number of unprinted diagonal or nondiagonal terms is very high, then the problem must be cleared and the difference  $b - \bar{b}$  reduced.

After completion of this check, the first part of the program using the command 1174 transmits control to the control program and the second part of the program is read out from the magnetic drum.

The second part of the program copies the program for computing the boundary function  $f(\alpha)$  from the magnetic drum, computes the boundary

values of  $\alpha_j = (j-1) \frac{2\pi}{250}$ ,  $j=1, 2, \dots, 250$ , the Fourier coefficients  $B_i$  from Formula (6), the coefficients  $C_i$  from Formula (7) and prints out the coefficients  $C_1, C_2, \dots, C_{24}$ .

After this, the second program seeks the harmonic function  $u(\xi, \eta)$  according to Formula (8) outside the ellipse  $s$ , and prints out these points and the solutions corresponding to them in the following sequence:  $\xi_1, \eta_1, u(\xi_1, \eta_1); \xi_2, \eta_2, u(\xi_2, \eta_2); \dots; \xi_n, \eta_n, u(\xi_n, \eta_n)$ , where  $n$  is the number of these points.

#### Operation of Console

The entire punchcard deck is assembled in the following order:

- (1) Basic program input card;
- (2) Blank punchcard;
- (3) Basic program;
- (4) Blank punchcard;
- (5) Initial data input card;
- (6) Blank punchcard;
- (7) Initial data;
- (8) Blank punchcard;
- (9) Initial data;
- (10) Blank punchcard;
- (11) Numerical data;
- (12) Blank punchcard;
- (13) Numerical data;

TABLE OF CONTROL STOPS

/119

Stop	Contents of instruction storage	Reason for stop	Action
0004 0204 0023 0032 0037	33 0007 — — 33 0002 — — 33 0003 — — 33 0004 — — 33 0005 — —	Incorrect program read-in. Incorrect input of numerical data.	Repeat read-in.
0150 0013 0223 1103 1125 1133 0301 0262 0351	33 0006 — — 33 0002 — — 33 0010 — — 33 0011 — — 33 0012 — — 33 0013 — — 33 0017 — — 33 0016 — — 33 0020 — —	Incorrect access to zero magnetic drum.      Incorrect access to first magnetic drum.	Repeat access to drum with key start.
0433 0073	33 0021 — — 33 0002 — —	End of computation. SP-0002 (block transfer 10→2).	
0153 0370	33 0013 — — 33 0010 — —	SP-0013 (computation of $\ln x$ ). SP-0010 (square root extraction).	
1155 1165	33 0014 — — 33 0015 — —	Nondiagonal term does not reach required accuracy. Diagonal term does not reach required accuracy.	Continue computation with key start.

For stop 33 0007 execute command 34 - - 0176 from console.

It may happen that when the machine is in good working condition it will often stop on commands 1155 and 1165 and print out the difference

$$1 - \sum_{j=1}^{250} \psi_i(\alpha_j) \psi_k(\alpha_j) > 10^{-4} \quad \text{when } i=k$$

$i=1, 2, \dots, 24$

or

$$\sum_{j=1}^{250} \psi_i(\alpha_j) \psi_k(\alpha_j) > 10^{-4} \quad \text{when } i \neq k.$$

$k=1, 2, \dots, i$

In this case, the problem must be cleared and the difference  $b - \bar{b}$  reduced in the initial data.

The results are printed out in the following sequence: the auxiliary points  $x_1^{(1)}, x_2^{(1)}; x_1^{(2)}, x_2^{(2)}; \dots; x_1^{(24)}, x_2^{(24)}$  are printed out, following  $x^{(24)}, x_3^{(24)}$  there will be a bit space and the coefficients  $C_1, C_2, \dots, C_{24}$  are printed out; without the space the coordinates of only the first designated point are printed out and the unknown solution corresponding to it, but the coordinates of the other points and the solutions corresponding to them are printed out with a space. This happens when the accuracy of orthonormalization is better than  $10^{-4}$ .

If any term, diagonal or nondiagonal, does not reach the required accuracy, then following the coefficients  $C_i, i = 1, \dots, 24$  there will be printed out

/120

$$1 - \sum_{j=1}^{250} \psi_i(\alpha_j) \psi_k(\alpha_j) > 10^{-4} \text{ when } i=k$$

$i=1, 2, \dots, i$

or

$$\sum_{j=1}^{250} \psi_i(\alpha_j) \psi_k(\alpha_j) > 10^{-4} \text{ when } i \neq k$$

$k=1, 2, \dots, i$

with a space. After these numbers (not reaching the accuracy), the coordinates of the designated points  $(\xi_i, \eta_i)$  and the solutions corresponding to them are printed out.

TABLE 12. MEMORY CONFIGURATION

Designation of data blocks	Beginning	End	Control sum
Basic program input card.	0001	0005	0005, 0050
Control program.	0006	0057	
Program for processing initial and numerical data.	0060	0346	
First program.	0060	1174	0044
Second program.	0220	0434	0047
Initial data.	0622	1332	0013, 0014
Numerical data.	0623	3665	0010, 0011

TABLE 13. MEMORY CONFIGURATION

Designation of blocks	Assignment in internal memory				Assignment on magnetic drum		Drum number
	During read-in		During operation				
	Beg.	End	Beg.	End	Beg.	End	
Basic program input card	0001	0005					
Control program	0006	0057					
Program for processing initial & numerical data	0060	0345					
First part of basic program	0346	1463	0060	1174	0000	1114	0
Working cells	2442	2471	2442	2471			
	1766	2357	1766	2357			
	2500	3071	2500	3071			
Results			3212	3665			
			3716	3775			
Second part of prog.	1464	1700	0220	0434	1115	1331	0
Working cells	0500	0511	0500	0511			
	1767	2360	1767				
Results			3666	3715			
b - b	0622	0622	0043	0043			
b	0623	0623	0051	0051			
Prog. for computing boundary function	0626	0626	0626	0626	1332	1332	
Orthonormalized func.		+(n-1)		+(n-1)		+(n-1)	
Numerical data, i.e., designated points			1766	2357	0000	23557	1
	0623	3662-n	0626	3665-n	2462	11521-n	

/121

### Set Up of Initial Data

The input card for the initial data has the following form:

30	0100		$n-1$
31		0622	0014
30	0100		$n-1$
31		0622	0013
35	0014	0013	0010
33	0022		
34			0001
77			0060
	0622	0002	0622
34			0134
00			
00			

where  $n$  is the length of the program for computing the boundary function  $f(\alpha)$ .

The initial information requires recording the following original data in the cells 0622 - 0625: 0622 — the difference  $b - \bar{b}$  in the decimal notation system, where  $b$  and  $\bar{b}$ , are the semi-minor axes of the confocal ellipses  $s$  and  $s_1$  respectively. 0623 -  $n$  for the third address — length of the program for computing the boundary function  $f(\alpha)$  in the octal notation system. In the second address of cell 0625 — the number of designated points  $N$  is written in the octal notation system. The number of designated points must not exceed  $[1420 - n/2]_8$ , if  $n$  is even;  $[1420 - (n-1)/2]_8$ , if  $n$  is odd.

The program for computing the boundary function must begin from cell 0626. The length of the program for computing the boundary function  $f(\alpha)$  must not exceed  $600|_{10}$  or  $1130|_8$ .

If in computing  $f(\alpha)$  the values of  $\cos x$  and  $\sin x$  are necessary, then they can be taken from cell 0020 and 0021. In compiling the programs the working cells used must certainly be taken into account in the length of the program for  $f(\alpha)$ .

If  $\cos x$  and  $\sin x$  are not necessary, cells 0020 and 0021 may be used as working cells. The numerical data must follow directly after the program for the boundary function  $f(\alpha)$ ; this is  $\xi_1, \eta_1; \xi_2, \eta_2; \dots; \xi_n, \eta_n$ .

TABLE 14			/122
Designation of the working block	Assignment in internal memory		
	Beginning	End	
A	0006	0057	
B	0060	0346	
C	0060	0377	
D	0400	0540	
E	0541	0611	
F	0612	1027	
G	1030	1121	
H	1122	1171	
I	0200	0314	
J	0315	0343	
K	0343	0433	

### FLOW DIAGRAM

#### Overall Flow Diagram

- A Control program block.
- B Block for processing initial and numerical data.
- C Standard program block.
- D Block for computing auxiliary points on confocal ellipse according to Formula (1).
- E Block for computing scalar products according to Formula (2).
- F Block for computing coefficients of orthonormalization according to Formulas (3<sub>1</sub>) and (3<sub>2</sub>).
- G Block for computing orthonormalized functions according to Formula (4).
- H Block for checking accuracy of orthonormalization.
- I Block for computing Fourier coefficients according to Formula (5).

- J Block for computing coefficients  $C_i$  according to Formula (6).
- K Block for finding unknown function outside ellipse according to Formula (7).

#### Control Program Block

- A 1 Record first part of basic program on zero magnetic drum.
- A 2 Record second part of basic program on zero magnetic drum.
- A 3 Read in initial data input card.
- A 4 Read in initial data.
- A 5 Read out first part of basic program from zero magnetic drum.
- A 6 Read out second part of basic program from zero magnetic drum.

#### Block for Processing Initial and Numerical Data

/123

- B 1 Standard program for converting array of numbers from decimal notation system to binary.
- B 2 Transfer  $b$  and  $b - \bar{b}$  in cells 0051 and 0043.
- B 3 Record program for computing boundary function on zero magnetic drum.
- B 4 Transfer numerical data, i. e., go to B 1 and record on zero magnetic drum.
- B 5 Go to A 5.

#### Standard Program Block

- C 1 Standard program for computing sine and cosine.
- C 2 Standard program for computing natural logarithm.
- C 3 Standard program for converting numbers from binary notation system to decimal with floating point.
- C 4 Standard program for computing specified integral by Simpson's method with automatic calling sequence.
- C 5 Standard program for square root extraction.

Block for Computing Auxiliary Points on Confocal Ellipse

- D 1 Compute  $\bar{a}$  and  $\bar{b}$  and transfer in cells 0050 and 0052.
- D 2 Clear counter for i.
- D 3 Convert from degree units to radians.
- D 4 Readdress i.
- D 5 Check if all degree units have been converted into radians; if yes go to D 6; if no go to D 3.
- D 6 Clear counter for i.
- D 7 Transfer argument in cell 0001.
- D 8 Go to C 1.
- D 9 Compute  $x_1^{(i)}, x_2^{(i)}$ .
- D 10 Transfer  $x_1^{(i)}, x_2^{(i)}$  in constant cells for storage.
- D 11 Clear counter for j.
- D 12 Transfer argument to cell 0001.
- D 13 Go to C 3.
- D 14 Print out  $j-x_1^{(i)}, x_2^{(i)}$ .
- D 15 Readdress j.
- D 16 Check if both  $x_1^{(i)}, x_2^{(i)}$  have been printed; if yes go to D 17; if no go to D 1.
- D 17 Readdress i.
- D 18 Retrieve j.
- D 19 Check if all  $x_1^{(i)}, x_2^{(i)}$  have been computed; if yes go to E 1; if no go to D 7.

Block for Computing Scalar Products

/124

- E 1 Transfer 1 - 1 - A for n.
- E 2 Transfer 1 - 1 - A for i.
- E 3 Transfer 1 - 1 - A for j.
- E 4 Readdress for n.
- E 5 Go to C 4.
- E 6 Clear output for computation by integral formula.

E 7 Transfer  $x_1^{(i)}, x_2^{(i)}$  in working cells.  
 E 8 Go to C 1.  
 E 9 Compute  $(x_1 - x_1^{(i)})^2 + (x_2 - x_2^{(i)})^2$ .  
 E 10 Transfer contents of 0001 to working cell.  
 E 11 Transfer argument 0001.  
 E 12 Go to C 2.  
 E 13 Compute  $\varphi_i$ .  
 E 14 Transfer result for storage.  
 E 15 Readdress for obtaining  $x_1^{(i)}, x_2^{(i)}$ .  
 E 16 Retrieve argument for using standard program C 4.  
 E 17 Go to E 8.  
 E 18 Output for computing under integral formula.  
 E 19 Multiply  $\varphi_i \varphi_k$ .  
 E 20 Output for standard program C 4.  
 E 21 Transfer result of integral computation for storage.  
 E 22 Readdress for j.  
 E 23 Check if all j have been computed; if yes go to E 3; if no go to C 4.  
 E 24 Readdress for i.  
 E 25 Check if all i have been computed; if yes go to E 25; if no go to E 3.  
 E 26 Go to F 1.

#### Block for Computing Coefficients of Orthonormalization

F 1 Transfer  $(\varphi_1, \varphi_1)$  to cell 0001.  
 F 2 Refer to C 5.  
 F 3 Compute  $A_{11} = \frac{1}{\sqrt{(\varphi_1, \varphi_1)}}$ .  
 F 4 Transfer constants (00 0002 000 0000) to k-counter.  
 F 5 Retrieve variable commands and set up constants for shaping.  
 F 6 Clear counter for j.  
 F 7 Shape commands for computing product  $A_{k-j,i}(\omega_k, \omega_i)$ .

- F 8 Compute product  $A_{k-j,i}(\omega_k \omega_i)$ .
- F 9 Compute Formula  $(3_4)$ .
- F 10 Check if all terms of Formula  $(3_4)$  have been computed; if yes go to F 11, if no go to F 14.
- F 11 Transfer value  $\alpha_{k,k-j}$  to storage. /125
- F 12 Readdress variable commands and constants.
- F 13 Check if all  $\alpha_{k,k-j}$  have been computed; if yes go to F 15; if no go to F 5.
- F 14 Set up shaping constants for computing next term and go to F 7.
- F 15 Set up constants for shaping.
- F 16 Clear counter for j.
- F 17 Shape commands for computing product  $\alpha_{k,k-i}, A_{k-i,j}$ .
- F 18 Compute product  $\alpha_{k,k-i}, A_{k-i,j}$ .
- F 19 Compute Formula  $(3_5)$ .
- F 20 Check if all terms  $B_{k,j}$  have been computed; if yes go to F 21; if no go to F 24.
- F 21 Transfer values  $B_{k,j}$  to storage.
- F 22 Readdress variable commands and constants.
- F 23 Check if all  $A_{k,j}$  have been computed; if yes go to F 25; if no go to F 17.
- F 24 Set up shaping constants for computing next term and go to F 17.
- F 25 Set up constants for shaping.
- F 26 Shape commands for computing the formula

$$\sum_{j=0}^{k-1} B_{k, k-j} \int_s \omega_{k-j}^2 ds_y. \quad (3_6)$$

- F 27 Compute product

$$B_{k, k-j} \int_s \omega_{k-j}^2 ds_y.$$

- F 28 Compute Formula  $(3_6)$ .

F 29 Check if all terms of Formula (3<sub>6</sub>) have been computed; if yes go to F 31; if no go to F 30.

F 30 Readdress variable commands for computing next term and go to F 27.

F 31 Shape commands for computation using the formula

$$2 \sum_{j=0}^{k-2} B_{h, k-j} \sum_{i=j+1}^{k-1} B_{h, i} \int_s \omega_{k-j} \omega_i ds_y. \quad (3_7)$$

F 32 Compute product

$$B_{h, k-j} B_{h, i} \int_s \omega_{k-j} \omega_i ds_y.$$

F 33 Compute Formula (3<sub>7</sub>).

F 34 Check if all terms of Formula (3<sub>7</sub>) have been computed; if yes go to /126 F 36; if no go to F 35.

F 35 Readdress variable commands for computing next term and go to F 32.

F 36 Compute radicand.

F 37 Refer to C 5.

F 38 Shape commands for computing Formula (1).

F 39 Compute Formula (1).

F 40 Set up constants for shaping.

F 41 Shape commands for computing Formula (2).

F 42 Transfer constants (00 0001 000 000) in i-counters.

F 43 Compute nondiagonal coefficient of given row by Formula (2).

F 44 Check if all nondiagonal coefficients of given row have been computed; if yes go to F 46; if no go to F 45.

F 45 Readdress commands for computing next nondiagonal coefficient and go to F 43.

F 46 Check if all coefficients  $A_{k,j}$  have been computed; if yes go to G 1; if no go to F 5.

#### Block for Computing Orthonormalized Functions

- G 1 Transfer some constants to working cells and clear counter for 1.
- G 2 Retrieve some commands and clear counter for k.
- G 3 Compute orthonormalized functions.
- G 4 Check if all orthonormalized functions have been computed for k; if yes go to G 5; if no go to G 2.
- G 5 Copy orthonormalized functions for k from operating memory on first magnetic drum.
- G 6 Readdress some commands for i.
- G 7 Check if all orthonormalized functions for k have been computed; if yes go to H 1; if no go to G 2.

#### Block for Checking Accuracy of Orthonormalization

- H 1 Transfer some commands in working cells.
- H 2 Clear counter for i.
- H 3 Transfer variable command to working cell.
- H 4 Clear counter for k.
- H 5 Read out orthonormalized coefficients for k from first magnetic drum.
- H 6 Retrieve variable command.
- H 7 Read out orthonormalized coefficients from first magnetic drum.
- H 8 Clear counter for j.
- H 9 Compute  $\sum_{j=1}^{250} \varphi_{kj} \varphi_{ij}$ ,  $i=1, 2, \dots, 24$ ;  $k=1, 2, \dots, i$ .
- H 10 Readdress for j.
- H 11 Check if cycle for j is completed; if yes go to 10; if not go to 7. /127
- H 12 Check accuracy of orthonormalization; has accuracy been reached; if yes go to 7; if no go to 11.
- H 13 Print out terms that have not reached the accuracy.
- H 14 Check if the cycle for k is completed; if yes go to 15; if no go to 7.

#### Block for Computing Fourier Coefficients from Formula (5)

- I 1 Read out program for computing boundary function  $f(\alpha)$  from zero

magnetic drum.

- I 2 Transfer argument in cell 0001 and go to C 1.
- I 3 Check if all  $f(\alpha_j)$ ,  $j = 1, \dots, 250$ ,  $\alpha_j = (j-1) \frac{2\pi}{250}$ , have been computed; if yes go to I 4; if no go to I 2.
- I 4 Read out orthonormalized coefficients from drum.
- I 5 Clear counter for  $k$ .
- I 6 Compute Fourier coefficients

$$B_k = \sum_{j=1}^{250} f(\alpha_j) \phi_{kj}, \quad k=1, 2, \dots, 24.$$

- I 7 Check if cycle for  $k$  is completed; if yes go to J 1; if no go to I 6.
- I 8 Clear counter for  $i$ .
- I 9 Clear working cell for variable command.
- I 10 Clear counter for  $k$ .
- I 11 Clear working cell for storage of sum.
- I 12 Compute  $C_i = \sum_{k=i}^{24} B_k A_{i,k}$ .
- I 13 Check if cycle for  $k$  is finished; if yes go to I 14, if no go to I 12.
- I 14 Transfer  $C_i$  for storage; go to C 3 and print out.
- I 15 Readdress variable command of retrieval and readdress for  $i$ .
- I 16 Check if cycle for  $i$  is finished; if yes go to I 17; if no go to I 9.
- I 17 Transfer designated points from magnetic drum.
- I 18 Clear counter for  $n$ .
- I 19 Transfer designated points in working cells.
- I 20 Clear counter for  $i$ .
- I 21 Has  $u(\xi_n, \eta_n) = \sum_{i=1}^{24} C_i w_i$  been computed?
- I 22 Check if cycle for  $i$  is finished; if yes go to I 23; if no go to I 2.

I 23 Print out  $u(\xi_n, \eta_n)$ .  
 I 24 Readdress for n.  
 I 25 Retrieve some commands for i.  
 I 26 Check if cycle for n is finished; if yes go to I 27; if no go to I 19.  
 I 27 Check if all  $2N_{\max}$  designated points have been computed; if yes go to I 27; if no go to I 19.  
 I 28 Retrieve some commands and go to readdress I 17.  
 I 29 Check if all designated points have been called; if yes go to I 30; if no go to I 18.  
 I 30 Stop.

### Control Example 1

Let us look at the following external Dirichlet boundary problem for the ellipse s:

$$x_1 = \cos \varphi, \quad x_2 = 0,8 \sin \varphi$$

under the boundary condition

$$u(\alpha) = \frac{x_1}{x_1^2 + x_2^2}, \quad 0 < \alpha < 2\pi$$

and seek the harmonic function outside the ellipse s.

In cell 0623 we transfer the value of the semi-minor axis  $b = 0.8$ . In cell 0622 we read-in the number  $b - \bar{b} = 0.2$ . The machine automatically selects the following values of the semiaxes for the auxiliary ellipse  $s_1$ :

$$x_1^{(i)} = 0,75 \cos \alpha_i, \quad x_2^{(i)} = 0,6 \sin \alpha_i, \quad i = 1, 2, \dots, 24.$$

In cell 0624 according to the third address, N is the length of the program for computing the boundary function  $f(\alpha)$ . 0625 according to the second address is the number of designated points - n (i. e.  $\xi_1, \eta_1; \xi_2, \eta_2; \dots; \xi_n, \eta_n$ ) in the octal notation system. In our case

$$N = 40 \Big|_{10} = 50 \Big|_8.$$

The approximate values for solution to this problem must be computed at the designated points in the region  $G_e$  outside the ellipse  $s$ . The values of the designated points are read in from cell 0623.

The precise solution to the problem has the form

$$u(\xi, \eta) = \frac{\xi}{\xi^2 + \eta^2} \quad (\xi, \eta) \in G_e.$$

Below we derive the initial and numerical data.

/129

Address		Commands				Comments
1		31	0100		0010	Initial data input card.
2		31		0622	0014	
3		30	0100		0010	
4		31		0622	0013	
5		35	0014	0013	0010	
6		33	0022			
7		34			0001	
0010		77			0060	
1			0622	0002	0622	
2		34			0134	
3		00				
4		00				
0622	+	0	+	2000	0000	b - $\bar{b}$ is difference between semiaxes. b is semi-minor axis of basic ellipse. n is length of program for computing boundary func.
3	+	0	+	8000	0000	
4					0005	
5				0050		The number of designated points in octal notation system.
6		03	0020	0020	0001	
7		03	0021	0021	0002	
0630		01	0001	0002	0001	
1		04	0020	0001	0001	
2		34			0272	

# NUMERICAL DATA

Address	Numbers				Address	Numbers			
0623	+	0	+	8000	0660	+	1	+	4600
4	+	1	+	1000	1	+	0	+	9000
5	+	0	+	8000	2	+	1	+	5600
6	+	1	+	1200	3	+	1	+	1200
7	+	0	+	8000	4	+	1	+	1200
0630	+	1	+	1400	5	+	1	+	1200
1	+	0	+	8000	6	+	1	+	2000
2	+	1	+	1600	7	+	1	+	1200
3	+	0	+	8000	0670	+	1	+	2800
4	+	1	+	1800	1	+	1	+	1200
5	+	0	+	8000	2	+	1	+	3600
6	+	1	+	2000	3	+	1	+	1200
7	+	0	+	8000	4	+	1	+	4400
0640	+	1	+	2400	5	+	1	+	1200
1	+	0	+	8000	6	+	1	+	5200
2	+	1	+	2600	7	+	1	+	1200
3	+	0	+	9000	0700	+	1	+	6000
4	+	1	+	1100	1	+	1	+	1200
5	+	0	+	9000	2	+	1	+	6800
6	+	1	+	1600	3	+	1	+	2000
7	+	0	+	9000	4	+	1	+	1300
0650	+	1	+	2100	5	+	1	+	2000
1	+	0	+	9000	6	+	1	+	2300
2	+	1	+	2600	7	+	1	+	2000
3	+	0	+	9000	0710	+	1	+	3300
4	+	1	+	3100	1	+	1	+	2000
5	+	0	+	9000	2	+	1	+	4300
6	+	1	+	3600	3	+	1	+	2000
7	+	0	+	9000	4	+	1	+	5300
5	+	1	+	2000	0730	+	1	+	3600
6	+	1	+	6300	1	+	1	+	5000
7	+	1	+	2000	2	+	1	+	4700
0720	+	1	+	7300	3	+	1	+	5000
1	+	1	+	2000	4	+	1	+	5800
2	+	1	+	8300	5	+	1	+	5000
3	+	1	+	5000	6	+	1	+	6900
4	+	1	+	1400	7	+	1	+	5000
5	+	1	+	5000	0740	+	1	+	8000
6	+	1	+	2500	1	+	1	+	5000
7	+	1	+	5000	2	+	1	+	9100

/130

The printed results of the control example are shown in Tables 15 and 16.

Table 15 gives the coordinates of the auxiliary points on the confocal ellipse in the following sequence:

$$x_1^{(1)}, x_2^{(1)}; \dots; x_1^{(23)}, x_2^{(23)}$$

and the value of the Fourier coefficients  $C_1, C_2, \dots, C_{24}$ .

Table 16 gives the coordinates of the designated points  $(\xi_i, \eta_i)$ , and the solutions  $u(\xi_i, \eta_i)$ , corresponding to them obtained on the machine, and the error

$$\epsilon_i = \left[ \frac{\xi}{\xi^2 + \eta^2} - u(\xi, \eta) \right], \quad i = 1, 2, \dots, 40.$$

The operating time of the machine for this example is 45 minutes. The distribution of designated points is shown on Figure 4.

TABLE 15

$i$	$x_1^{(i)}$	$x_2^{(i)}$	$C_i$
1	-0.10971885 · 10 <sup>-1</sup>	0.60000000	-0.22206542 · 10 <sup>-5</sup>
2	0.32915656 · 10 <sup>-1</sup>	-0.60000000	0.20515213 · 10 <sup>-5</sup>
3	0.75000000	0.00000000	-0.76608258 · 10 <sup>-1</sup>
4	-0.75000000	-0.17555016 · 10 <sup>-1</sup>	0.76589811 · 10 <sup>-1</sup>
5	-0.53033006	-0.42426408	0.10152326
6	0.53033008	0.42426407	-0.10152297
7	0.53033011	-0.42426404	-0.10152337
8	-0.53033009	0.42426406	0.10152883
9	0.64951907	-0.29999997	-0.90643412 · 10 <sup>-1</sup>
10	-0.64951906	0.29999998	0.90631226 · 10 <sup>-1</sup>
11	-0.37500001	0.51961523	0.99701809 · 10 <sup>-1</sup>
12	0.37500003	-0.51961522	-0.99703552 · 10 <sup>-1</sup>
13	0.37499999	0.51961524	-0.99706374 · 10 <sup>-1</sup>
14	-0.37499997	0.51961525	0.99706651 · 10 <sup>-1</sup>
15	0.64951905	0.30000000	-0.90641500 · 10 <sup>-1</sup>
16	-0.64951904	-0.30000001	0.90637014 · 10 <sup>-1</sup>
17	0.19411427	0.57955549	0.66432247 · 10 <sup>-1</sup>
18	-0.19411425	-0.57955550	0.66432366 · 10 <sup>-1</sup>
19	-0.19411429	0.57955549	0.66436770 · 10 <sup>-1</sup>
20	0.19411431	0.57955549	-0.66436330 · 10 <sup>-1</sup>
21	0.72444437	0.15529142	-0.80437784 · 10 <sup>-1</sup>
22	-0.72444436	-0.15529144	0.80449894 · 10 <sup>-1</sup>
23	-0.72444437	0.15529141	0.80453934 · 10 <sup>-1</sup>
24	0.72444438	-0.15529139	-0.80435528 · 10 <sup>-1</sup>

TABLE 16

$n$	$\xi_n$	$\eta_n$	$u(\xi_n \eta_n)$	$\varepsilon_n$
1	0.80000000	0.10000000	0.48780491	$-0.4 \cdot 10^{-7}$
2	0.80000000	0.12000000	0.38461539	$-0.1 \cdot 10^{-7}$
3	0.80000000	$0.13999999 \cdot 10^1$	0.30769231	$-0.1 \cdot 10^{-7}$
4	0.80000000	$0.16000000 \cdot 10^1$	0.25000001	$-0.1 \cdot 10^{-7}$
5	0.80000000	$0.17999999 \cdot 10^1$	0.20618559	$-0.3 \cdot 10^{-7}$
6	0.80000000	$0.20000000 \cdot 10^1$	0.17241383	$-0.4 \cdot 10^{-7}$
7	0.80000000	$0.24000000 \cdot 10^1$	0.12500006	$-0.6 \cdot 10^{-7}$
8	0.80000000	$0.25999999 \cdot 10^1$	0.10810818	$-0.8 \cdot 10^{-7}$
9	0.90000000	$0.11000000 \cdot 10^1$	0.44554459	$-0.4 \cdot 10^{-7}$
10	0.90000000	$0.16000000 \cdot 10^1$	0.26706233	$-0.2 \cdot 10^{-7}$
11	0.90000000	$0.20999999 \cdot 10^1$	0.17241384	$-0.5 \cdot 10^{-7}$
12	0.90000000	$0.25999999 \cdot 10^1$	0.11889043	$-0.8 \cdot 10^{-7}$
13	0.90000000	$0.30999999 \cdot 10^1$	$0.86372452 \cdot 10^{-1}$	$-0.92 \cdot 10^{-6}$
14	0.90000000	$0.35999999 \cdot 10^1$	$0.65359581 \cdot 10^{-1}$	$-0.104 \cdot 10^{-6}$
15	0.90000000	$0.45999999 \cdot 10^1$	$0.40965076 \cdot 10^{-1}$	$-0.124 \cdot 10^{-6}$
16	0.90000000	$0.55999999 \cdot 10^1$	$0.27976515 \cdot 10^{-1}$	$-0.14 \cdot 10^{-6}$
17	$0.12000000 \cdot 10^1$	$0.12000000 \cdot 10^1$	0.41666670	$-0.4 \cdot 10^{-7}$
18	$0.12000000 \cdot 10^1$	$0.20000000 \cdot 10^1$	0.22058829	$-0.6 \cdot 10^{-7}$
19	$0.12000000 \cdot 10^1$	$0.27999999 \cdot 10^1$	0.12931042	$-0.8 \cdot 10^{-7}$
20	$0.12000000 \cdot 10^1$	$0.35999999 \cdot 10^1$	$0.83333438 \cdot 10^{-1}$	$-0.105 \cdot 10^{-6}$
21	$0.12000000 \cdot 10^1$	$0.44000000 \cdot 10^1$	$0.57692429 \cdot 10^{-1}$	$-0.102 \cdot 10^{-6}$
22	$0.12000000 \cdot 10^1$	$0.51999999 \cdot 10^1$	$0.42134966 \cdot 10^{-1}$	$-0.135 \cdot 10^{-6}$
23	$0.12000000 \cdot 10^1$	$0.60000000 \cdot 10^1$	$0.32051428 \cdot 10^{-1}$	$-0.146 \cdot 10^{-6}$
24	$0.12000000 \cdot 10^1$	$0.68000000 \cdot 10^1$	$0.25167941 \cdot 10^{-1}$	$-0.156 \cdot 10^{-6}$
25	$0.20000000 \cdot 10^1$	$0.12999999 \cdot 10^1$	0.35149391	$-0.7 \cdot 10^{-7}$
26	$0.20000000 \cdot 10^1$	$0.22999999 \cdot 10^1$	0.21528533	$-0.8 \cdot 10^{-7}$
27	$0.20000000 \cdot 10^1$	$0.32999999 \cdot 10^1$	0.13431844	$-0.11 \cdot 10^{-7}$
28	$0.20000000 \cdot 10^1$	$0.43000000 \cdot 10^1$	$0.88928536 \cdot 10^{-1}$	$-0.124 \cdot 10^{-6}$
29	$0.20000000 \cdot 10^1$	$0.53000000 \cdot 10^1$	$0.6224850 \cdot 10^{-1}$	$-0.139 \cdot 10^{-6}$
30	$0.20000000 \cdot 10^1$	$0.63000000 \cdot 10^1$	$0.45777217 \cdot 10^{-1}$	$-0.152 \cdot 10^{-6}$
31	$0.20000000 \cdot 10^1$	$0.73000000 \cdot 10^1$	$0.34910269 \cdot 10^{-1}$	$-0.162 \cdot 10^{-6}$
32	$0.20000000 \cdot 10^1$	$0.83000000 \cdot 10^1$	$0.27438778 \cdot 10^{-1}$	$-0.172 \cdot 10^{-6}$
33	$0.50000000 \cdot 10^1$	$0.13999999 \cdot 10^1$	0.18546007	$-0.13 \cdot 10^{-6}$
34	$0.50000000 \cdot 10^1$	$0.25000000 \cdot 10^1$	0.16000013	$-0.13 \cdot 10^{-6}$
35	$0.50000000 \cdot 10^1$	$0.35999999 \cdot 10^1$	0.13171774	$-0.15 \cdot 10^{-6}$
36	$0.50000000 \cdot 10^1$	$0.46999999 \cdot 10^1$	0.10617981	$-0.16 \cdot 10^{-6}$
37	$0.50000000 \cdot 10^1$	$0.58000000 \cdot 10^1$	$0.85266193 \cdot 10^{-1}$	$-0.163 \cdot 10^{-6}$
38	$0.50000000 \cdot 10^1$	$0.69000000 \cdot 10^1$	$0.68861210 \cdot 10^{-1}$	$-0.172 \cdot 10^{-6}$
39	$0.50000000 \cdot 10^1$	$0.80000000 \cdot 10^1$	$0.56179955 \cdot 10^{-1}$	$-0.18 \cdot 10^{-6}$
40	$0.50000000 \cdot 10^1$	$0.91000000 \cdot 10^1$	$0.46378075 \cdot 10^{-1}$	$-0.188 \cdot 10^{-6}$

/131

Control Example 2

Let us look at the external Dirichlet boundary problem for the ellipse

S

$$x_1 = \cos \varphi, \quad x_2 = 0.5 \sin \varphi.$$

Under the boundary condition

$$u(\alpha) = \frac{-2x_1x_2}{[x_1^2 + x_2^2]^2}, \quad 0 \leq \alpha \leq 2\pi$$

we seek the harmonic function outside the ellipse  $s$ .

In cell 0623, we transfer the value for the semi-minor axis  $b = 0.5$ . In cell 0622 we read in the number  $b - \bar{b} = 0.1$ . The machine selects the following values of the semiaxes for the auxiliary ellipse  $s_1$ :

$$x_1^{(i)} = 0,8 \cos \alpha_i, \quad x_2^{(i)} = 0,4 \sin \alpha_i, \quad i = 1, 2, \dots, 24.$$

In this case, the number of designated points is  $i = {}^{43}_{10} = {}^{53}_8$ .

/132

The exact solution to this problem has the form

$$u(\xi, \eta) = \frac{-2\xi\eta}{[\xi^2 + \eta^2]^2}, \quad (\xi, \eta) \in G_e.$$

Below we give the initial and numerical data.

Address						Command					
1		30	0100		0014						
2		31		0622	0014						
3		30	0100		0014						
4		31		0622	0013						
5		35	0014		0010						
6		33	0022								
7		34			0001						
0010		77			0060						
1			0622	0002	0622						
2		34			0134						
3		00									
4		00									
					0622		+	0	+	1000	0000
					3		+	0	+	5000	0000
					4						0011
					5					0053	
					6			03	0020	0020	0001
					7			03	0021	0021	0002
					0630			01	0001	0002	0001
					1			03	0001	0001	0001
					2			03	0020	0021	0002
					3			03	0636	0002	0002
					4			04	0002	0001	0001
					5			34			0272
					6			02	3000		

NUMERICAL DATA											
Address						Command					
0000	+	0	+	1000		0040	+	0	+	5000	
1	+	0	+	8500		1	+	1	+	4000	
2	+	0	+	1000		2	+	0	+	5000	
3	+	0	+	9000		3	+	1	+	4500	
4	+	0	+	1000		4	+	0	+	7000	

# NUMERICAL DATA (con't)

Address	Numbers				Address	Numbers			
5	+	0	+	9500	5	+	0	+	8000
6	+	0	+	1000	6	+	0	+	7000
7	+	1	+	1000	7	+	0	+	9000
0010	+	0	+	1000	0050	+	0	+	7000
1	+	1	+	1050	1	+	1	+	1500
2	+	0	+	1000	2	+	0	+	7000
3	+	1	+	1100	3	+	1	+	1700
4	+	0	+	4000	4	+	0	+	7000
5	+	1	+	1200	5	+	1	+	1900
6	+	0	+	4000	6	+	0	+	7000
7	+	1	+	1400	7	+	1	+	2100
0020	+	0	+	4000	0060	+	1	+	1000
1	+	1	+	1600	1	+	0	+	8000
2	+	0	+	4000	2	+	1	+	1000
3	+	1	+	1800	3	+	1	+	1100
4	+	0	+	4000	4	+	1	+	1000
5	+	1	+	2000	5	+	1	+	1300
6	+	0	+	4000	6	+	1	+	1000
7	+	1	+	2200	7	+	1	+	2300
0030	+	0	+	5000	0070	+	1	+	1000
1	+	0	+	8500	1	+	1	+	2400
2	+	0	+	5000	2	+	1	+	1000
3	+	1	+	2500	3	+	1	+	2600
4	+	0	+	5000	4	+	1	+	1500
5	+	1	+	3000	5	+	0	+	8000
6	+	0	+	5000	6	+	1	+	1500
7	+	1	+	3500	7	+	1	+	3000
0100	+	1	+	1500	3	+	0	+	9000
1	+	1	+	3700	4	+	1	+	2000
2	+	1	+	1500	5	+	1	+	1000
3	+	1	+	4400	6	+	1	+	2000
4	+	1	+	1500	7	+	1	+	2000
5	+	1	+	5100	0120	+	1	+	2000
6	+	1	+	1500	1	+	1	+	3000
7	+	1	+	5800	2	+	1	+	2000
0110	+	1	+	2000	3	+	1	+	400
1	+	0	+	8000	4	+	1	+	2000
2	+	1	+	2000	5	+	1	+	5000

/133

TABLE 17

$i$	$x_1^{(i)}$	$x_2^{(i)}$	$c_i$
1	$-0.11703344 \cdot 10^{-7}$	0.40000000	$0.96853857 \cdot 10^{-6}$
2	$-0.35110033 \cdot 10^{-7}$	$-0.40000000$	$-0.16354052 \cdot 10^{-5}$
3	0.80000000	0.00000000	$-0.29542828 \cdot 10^{-4}$
4	$-0.80000000$	$-0.11703344 \cdot 10^{-7}$	$-0.20274971 \cdot 10^{-4}$
5	$-0.56568540$	$-0.28284272$	0.17703348
6	$-0.56568542$	0.28284271	0.17703663
7	0.56568545	$-0.28284269$	$-0.17703063$
8	$-0.56568543$	0.28284270	$-0.17702967$
9	0.69282034	$-0.19999998$	$-0.38299288 \cdot 10^{-1}$
10	$-0.69282033$	0.19999999	$-0.38298252 \cdot 10^{-1}$
11	$-0.40000001$	0.34641015	$-0.65954406$
12	0.40000003	$-0.34641015$	$-0.65954315$

TABLE 17 (con't)

$i$	$x_i^{(i)}$	$x_2^{(i)}$	$C_i$
13	0.39999999	0.34641016	0.65954371
14	-0.39999997	-0.34641017	0.65954367
15	0.69282032	0.20000000	0.38276880 · 10 <sup>-1</sup>
16	-0.69282031	-0.20000001	0.38283523 · 10 <sup>-1</sup>
18	0.20705522	0.38637033	0.15523954 · 10 <sup>1</sup>
18	-0.20705520	-0.38637033	0.15523975 · 10 <sup>1</sup>
19	-0.20705524	0.38637032	-0.15523971 · 10 <sup>1</sup>
20	0.20705527	-0.38637032	-0.15523962 · 10 <sup>1</sup>
21	0.77274066	0.10352761	0.97296862 · 10 <sup>-2</sup>
32	-0.17274065	-0.10352763	0.97203732 · 10 <sup>-2</sup>
23	0.77274066	0.10352760	0.96886752 · 10 <sup>-2</sup>
24	0.77274067	-0.10352759	-0.96832523 · 10 <sup>-2</sup>

TABLE 18

$i$	$\xi_i$	$\eta_i$	$u(\xi_i, \eta_i)$	$\varepsilon_i$
1	0.10000000	0.85000000	-0.31584753	0.101214 · 10 <sup>-2</sup>
2	0.10000000	0.90000000	-0.26709592	0.60187 · 10 <sup>-3</sup>
3	0.10000000	0.95000000	-0.22780316	0.38224 · 10 <sup>-3</sup>
4	0.10000000	0.10000000 · 10 <sup>1</sup>	-0.19580721	0.25200 · 10 <sup>-3</sup>
5	0.10000000	0.10499999 · 10 <sup>1</sup>	-0.16950372	0.16182 · 10 <sup>-3</sup>
6	0.10000000	0.11000000 · 10 <sup>1</sup>	-0.14768900	0.12072 · 10 <sup>-3</sup>
7	0.40000000	0.12000000 · 10 <sup>1</sup>	-0.37490475	0.9525 · 10 <sup>-4</sup>
8	0.40000000	0.13999999 · 10 <sup>1</sup>	-0.24914741	0.5159 · 10 <sup>-4</sup>
9	0.40000000	0.16000000 · 10 <sup>1</sup>	-0.17298143	0.2895 · 10 <sup>-4</sup>
10	0.40000000	0.17999999 · 10 <sup>1</sup>	-0.12455015	0.1732 · 10 <sup>-4</sup>
11	0.40000000	0.20000000 · 10 <sup>1</sup>	-0.92444604 · 10 <sup>-1</sup>	0.11017 · 10 <sup>-4</sup>
12	0.40000000	0.22000000 · 10 <sup>1</sup>	-0.70392605 · 10 <sup>-1</sup>	0.7395 · 10 <sup>-4</sup>
13	0.50000000	0.85000000	-0.89885234	-0.66 · 10 <sup>-6</sup>
14	0.50000000	0.25000000 · 10 <sup>1</sup>	-0.59166383 · 10 <sup>-1</sup>	0.5214 · 10 <sup>-4</sup>
15	0.50000000	0.30000000 · 10 <sup>1</sup>	-0.35059446 · 10 <sup>-1</sup>	0.2643 · 10 <sup>-4</sup>
16	0.50000000	0.35000000 · 10 <sup>1</sup>	-0.22398476 · 10 <sup>-1</sup>	0.1524 · 10 <sup>-4</sup>
17	0.50000000	0.40000000 · 10 <sup>1</sup>	-0.15146966 · 10 <sup>-1</sup>	0.963 · 10 <sup>-5</sup>
18	0.50000000	0.45000000 · 10 <sup>1</sup>	-0.10707261 · 10 <sup>-1</sup>	0.650 · 10 <sup>-5</sup>
19	0.70000000	0.80000000	-0.87714309	-0.1881 · 10 <sup>-4</sup>
20	0.70000000	0.90000000	-0.74555695	0.518 · 10 <sup>-5</sup>
21	0.70000000	0.15000000 · 10 <sup>1</sup>	-0.27968637	0.3018 · 10 <sup>-4</sup>
22	0.70000000	0.17000000 · 10 <sup>1</sup>	-0.20830399	0.2204 · 10 <sup>-4</sup>
23	0.70000000	0.19000000 · 10 <sup>1</sup>	-0.15822332	0.1572 · 10 <sup>-4</sup>
24	0.70000000	0.20999999 · 10 <sup>1</sup>	-0.12243752	0.1146 · 10 <sup>-4</sup>
25	0.10000000 · 10 <sup>1</sup>	0.80000000	-0.59488382	0.17 · 10 <sup>-6</sup>
26	0.10000000 · 10 <sup>1</sup>	0.10000000 · 10 <sup>1</sup>	-0.45042692	0.1430 · 10 <sup>-4</sup>
27	0.10000000 · 10 <sup>1</sup>	0.12999999 · 10 <sup>1</sup>	-0.35929066	0.1891 · 10 <sup>-4</sup>
28	0.10000000 · 10 <sup>1</sup>	0.22999999 · 10 <sup>1</sup>	-0.11625829	0.873 · 10 <sup>-5</sup>
29	0.10000000 · 10 <sup>1</sup>	0.24000000 · 10 <sup>1</sup>	-0.10503055	0.778 · 10 <sup>-5</sup>
30	0.10000000 · 10 <sup>1</sup>	0.25999999 · 10 <sup>1</sup>	-0.86347263 · 10 <sup>-1</sup>	0.6228 · 10 <sup>-4</sup>
31	0.15000000 · 10 <sup>1</sup>	0.80000000	-0.28734960	0.328 · 10 <sup>-5</sup>
32	0.15000000 · 10 <sup>1</sup>	0.30000000 · 10 <sup>1</sup>	-0.71106808 · 10 <sup>-1</sup>	0.4303 · 10 <sup>-5</sup>
33	0.15000000 · 10 <sup>1</sup>	0.37000000 · 10 <sup>1</sup>	-0.43683835 · 10 <sup>-1</sup>	0.2573 · 10 <sup>-6</sup>
34	0.15000000 · 10 <sup>1</sup>	0.44000000 · 10 <sup>1</sup>	-0.28264387 · 10 <sup>-1</sup>	0.1615 · 10 <sup>-6</sup>

/134

TABLE 18 (con't)

i	$\xi_i$	$\eta_i$	$u(\xi_i, \eta_i)$	$\varepsilon_i$
35	0,15000000 · 10 <sup>1</sup>	0,50999999 · 10 <sup>1</sup>	-0,19156800 · 10 <sup>-1</sup>	0,1065 · 10 <sup>-5</sup>
36	0,15000000 · 10 <sup>1</sup>	0,58000000 · 10 <sup>1</sup>	-0,13507610 · 10 <sup>-1</sup>	0,740 · 10 <sup>-5</sup>
37	0,20000000 · 10 <sup>1</sup>	0,80000000	-0,14862975	0,283 · 10 <sup>-5</sup>
38	0,20000000 · 10 <sup>1</sup>	0,90000000	-0,15559766	0,332 · 10 <sup>-5</sup>
39	0,20000000 · 10 <sup>1</sup>	0,10000000 · 10 <sup>1</sup>	-0,15999619	0,381 · 10 <sup>-5</sup>
40	0,20000000 · 10 <sup>1</sup>	0,20000000 · 10 <sup>1</sup>	-0,12499422	0,578 · 10 <sup>-5</sup>
41	0,20000000 · 10 <sup>1</sup>	0,30000000 · 10 <sup>1</sup>	-0,71002050 · 10 <sup>-1</sup>	0,3867 · 10 <sup>-5</sup>
42	0,20000000 · 10 <sup>1</sup>	0,40000000 · 10 <sup>1</sup>	-0,39997789 · 10 <sup>-1</sup>	0,2211 · 10 <sup>-5</sup>
43	0,20000000 · 10 <sup>1</sup>	0,50000000 · 10 <sup>1</sup>	-0,23779920 · 10 <sup>-1</sup>	0,1292 · 10 <sup>-5</sup>

The printed results of the control example are shown on Tables 17 and 18.

Table 17 gives the coordinates of the auxiliary points on the confocal ellipse and the value of the Fourier coefficients.

Table 18 gives the coordinates of the designated points  $(\xi_i, \eta_i)$  and the solutions corresponding to them  $u(\xi_i, \eta_i)$ , obtained on the machine and error

$$\varepsilon_i = \left[ \frac{-2\xi_i\eta_i}{[\xi_i^2 + \eta_i^2]^2} u(\xi_i, \eta_i) \right], \quad i=1,2,\dots,43.$$

The machine time for this example is 50 minutes.

The distribution of designated points is shown on Figure 5.

#### OPERATIONAL CODE FOR THE METHOD OF GENERALIZED FOURIER SERIES

Address	Command					Address	Command				
0000		00				0010		30	0400		1114
1		30	0100		1664	1		71			0050
2		31		0014	0050	2		35	0044	0050	0015
3		35	0005	0050	0006	3		33	0002		
4		33	0001			4		34			0006
5		67	0077	2357	0426	5		30	1400	1115	0214
6		30	1400		1114	6		31		1464	0047
7		31		0347	0044	7		30	0400	1115	0214

Address				Command				Address				Command			
0020		71		0050	0120		41	0004	0132	0004					
1		35	0047	0050	0024	1	36	0125	0004	0072					
2		33	0003			2	33		0002	0003					
3		34			0015	3	35			0074					
4		30	0100		0013	4				0001					
5		31		0001		5				3777					
6		34			0001	6	37		3777						
7		30	0400		1114	7	04	1200							
0030		31		0060	0010	0130		3700							
1		35	0044	0010	0505	0		1100							
2		33	0004			2		2040							
3		34			0027	3	34	1463	0631	2315					
4		30	0400	1115	0214	4	14		0350	0351					
5		31		0220	0010	5		0622		0043					
6		35	0047	0010	0256	6		0623		0051					
7		33	0005			7		0624		0042					
0040		34		0034		0140	62	0042	0124	0352					
1		00				1	22	0323	0352	0143					
2		00				2	22	0324	0352	0145					
3		00				3	00								
4		00				4	31		0626	0046					
5		00				5	00								
6		00				6	71		0625	0010					
7		00				7	35	0046	0010	0152					
0050		00				0150	33	0006							
1		00				1	34			0143					
2		00				2		0145		0054					
3		00				3	22	0321	0042	0352					
4		00				4	62	0327	0352	0352					
5		00				5	17	0352	0124	0353					
6		00				6	35	0353		0160					
7		00				7	62	0352	0124	0352					
0060		34		0002	0061	0160		0352		0055					
1		26	0002	0026	0063	1	26	0352	0013	0353					
2		22	0124	0063	0063	2	22	0353	0322	0057					
3		33		0002	0001	3	17	0625	0325	0056					
4		17	0125	0001	0004	4	17	0625	0125	0354					
5		20	0001	0004	0076	5	75	0354		0237					
6		22	0124	0076	0076	6	17	0625	0326	0354					
7		22	0125	0004	0001	7	22	0354	0354	0354					
0070		22	0126	0001	0122	0170	26	0354	0113	0354					
1		22	0002	0124	0063	1	36	0354	0352	0232					
2		76	0001	0133	0117	2	62	0354	0352	0354					
3		33		0002		3		0124		0351					
4		17	0126	0076	0001	4	62	0352	0124	0355					
5		35		0001	0063	5	22	0330	0355	0177					
6		33		0002	0002	6		0177		0201					
7		22	0076	0126	0076	7	00								
0100		22	0124	0122	0122	0200	31		0623	0010					
1		45		0001	0002	1	00								
2		26	0002	0106	0004	2	31		0623	0011					
3		03	0127	0002	0002	3	35	0010	0011	0206					
4		17	0130	0001	0003	4	33	0007							
5		76	0131	0003	0073	5	34			0177					
6		41	0132	0004	0004	6	22	0331	0353	0210					
7		01	0003	0002	0002	7	77			0060					
0110		66	0001	0004	0001	0210	00								
1		76		0001	0103	1	00								
2		25	0002	0004	0001	2	22	0332	0355	0215					
3		36	0132	0004	0121	3	22	0334	0352	0217					
4		04	0001	0127	0001	4	22	0335	0352	0220					
5		42	0004	0132	0004	5	00								
6		35			0113	6	31		0623	0010					
7		03	0001	0127	0001	7	00								

/135

Address					Command					Address					Command				
0220		00								0320					34				0311
1		31		0623	0010					1								0626	
2		35	0010	0333	0225					2									
3		33	0010							3					30	1400	0001	1332	
4		34			0215					4				30	0400	1332			
5		35	0350	0351	0027					5					3777				
6		22	0335	0057	0335					6									
7		22	0045	0124	0045					7						3777			3666
0230		00								0330					30	0100			
1		34			0171					1						0623			0623
2					0351					2					30	1403			
3			0354		0352					3					77	3777	3777		3777
4		26	0354	0013	0353					4					20	0010	0333		0623
5			0352		0053					5					30	1400	2462		
6		34			0174					6									0003
7			0336		0352					7					01	1000			
0240		26	0336	0013	0353					0340									0002
1		34		0211	0174					1					34				0265
2					0352					2					34				0270
3		22	0216	0344	0216					3					34				0276
4		22	0334	0336	0334					4							0003		
5			0216		0221					5									0626
6			0625		0001					6						0003			0627
7		25	0001	0077	0353					7					04	0001	0126		0004
0250		02	0337	0353	0337					0350					14	0004	0003		0004
1			0623		0002					1					41	0004	0130		0004
2			0002		0003					2					26	0004	0046		0002
3			0341		0311					3					35	0002			0066
4			0624		0004					4					02	0127	0003		0003
5			0004		0005					5					26	0004	0036		0005
6		03	0005	0005	0006					6					03	0003	0121		0004
7		03	0051	0051	0007					7					03	0004	0004		0003
0260		04	0006	0007	0006					0360					01	0003	0122		0002
1		03	0003	0003	0007					1					03	0002	0002		0002
2		01	0006	0007	0006					2					01	0002	0003		0003
3		36	0006	0337	0303					3					01	0003	0123		0003
4		00								4					01	0002	0124		0002
5		01	0005	0001	0005					5					03	0002	0003		0002
6		34		0264	0256					6					02	0125	0002		0003
7			0342		0311					7					04	0004	0003		0002
0270		02	0004	0001	0004					0370					25	0002	0077		0003
1		34		0264	0255					1					03	0003	0003		0004
2		00								2					01	0127	0004		0003
3		01	0003	0001	0003					3					04	0002	0003		0002
4		34		0264	0253					4					02	0127	0004		0004
5			0343		0272					5					04	0004	0003		0003
6		02	0002	0001	0002					6					22	0117	0005		0004
7		34		0264	0252					7					76	0004	0120		0114
0300					0351					0400						0002			0004
1			0352		0053					1						0003			0002
2		34			0313					2						0004			0003
3			0003		0626					3					15	0002	0005		0002
4			0005		0627					4					22	0005	0127		0004
5		22	0303	0340	0303					5					15	0003	0004		0003
6		22	0304	0340	0304					6					37	0400			
7		22	0352	0340	0352					7					37	0600			
0310		35	0352	0055	0312					0410					75	1552	3107		0735
1		00								1					75	3117	0500		1322
2			0124		0351					2					74	3210	3556		1027
3		62	0352	0124	0355					3						1322	0647		1264
4		34		0330	0212					4					75	1621	2273		2707
5					0352					5						1444	0773		1242
6			0346		0303					6					01	1000			
7			0347		0304					7						1000			

/136

Address		Command				Address		Command			
0420		36	0001	0162	0153	0520		00			
1		06	0001	0001	0002	1		34		0231	0223
2		01	0002	0163	0003	2		22	0340	0327	0340
3		02	0002	0163	0004	3		34		0231	0221
4		04	0004	0003	0003	4		17	0342	0330	0342
5		43	0003	0003	0002	5		26	0342	0126	0313
6		43	0155	0002	0004	6		16	0321	0313	0253
7		41	0004	0156	0004	7		16	0342	0313	0313
0430		43	0004	0002	0004	0530		16	0322	0313	0313
1		41	0004	0157	0004	1		22	0340	0327	0340
2		43	0004	0002	0004	2		16	0340	0323	0244
3		01	0004	0160	0004	3		00			
4		03	0004	0003	0002	4			0251		0263
5		07	0001		0004	5			0251		0274
6		02	0004	0161	0004	6		22	0341	0331	0341
7		03	0004	0164	0004	7			0334		0001
0440		01	0004	0002	0002	0540		00			
1		35			0154	1			0001		0340
2		33		0013		2		00			
3		37				3		02	0335	0334	0342
4		12		1150	0200	4		36	0336	0342	0260
5		06	0006	1450	0074	5			0342		0336
6		02	0252	2526	3155	6			0341		0314
7		01	1777	3777	3775	7		03	0337	0336	0225
0450			1000			0550		01	0334	0336	0231
1		41	1000			1			0231		0001
2			1324	0236	1464	2		00			
3				3102	3770	3		01	0340	0001	0244
4		14		0004	0002	4			0001		0340
5		15	0001	0001	0003	5		06	0336	0264	0342
6		76	0211	0003	0173	6		03	0244	0342	0222
7		03	0003	0210	0003	7		06	0336	0264	0342
0460		76	0003	0211	0210	0560		01	0334	0342	0251
1		65		0067	0004	1			0220		0253
2		65	0004	0001	0004	2			0251		0001
3		76	0003	0212	0201	3		60			
4		04	0003	0213	0003	4		01	0253	0001	0253
5		35			0173	5		01	0251	0336	0251
6		26	0002	0004	0002	6		36	0251	0231	0273
7		03	0003	0212	0003	7		05	0253	0276	0253
0470		14	0003	0003	0005	0570		01	0244	0253	0244
1		22	0002	0005	0002	1		01	0244	0253	0253
2		76	0002	0214	0177	2		04	0336	0332	0342
3		47	0004		0004	3		03	0253	0342	0253
4		26	0004	0006	0004	4		06	0336	0302	0336
5		55	0004	0001	0004	5		02	0253	0222	0342
6		22	0004	0002	0002	6		76	0253	0333	0311
7		37	0734	3262	2000	7		04	0342	0253	0342
0500		01	1000			0600			0253		0222
1		04	1200			1		76	0225	0342	0270
2		04	1177	3777	3777	2		00			
3			0100			3		00			
4		34		0341	0216	4			0231		0334
5		26	0341	0026	0340	5		05	0336	0304	0336
6		14	0324	0225	0314	6		34			0254
7		40	0325		0231	7					0342
0510		16	0340	0320	0222	0610			0220		
1		00				1		01		0222	
2		17	0342	0330	0244	2					0251
3		16	0244	0225	0225	3					0334
4		00				4		74		0231	0223
5		62	0225	0244	0225	5					0001
6		22	0225	0326	0225	6			0001		
7		26	0342	0013	0342	7			3777		

/137

Address				Command				Address				Command			
0620					0003			0720							
1	03	1400						1	05	1700					
2	01	1000						2	10	1510					
3	00							3	07	1130					
4	00							4	10	1774					
5	00							5	07	1510					
6	00							6	11	1072					
7	00							7	04	1700					
0630	00							0730	10	1414					
1	00							1	10	1224					
2	25		0040	0002				2	11	1262					
3	76	0001	0346	0371				3					0002		
4	06	0001	0001	0005				4		2452			0001		
5	41	0005	0374	0003				5	77	3777	3777		3777		
6	36	0375	0005	0351				6				0001			
7	42	0003	0376	0003				7	03	0002		3212	2462		
0640	17	0001	0377	0005				0740	73	1467		2145	2625		
1		0375		0004				1	03	1441		0131	0455		
2	36	0005	0373	0355				2				0373			
3		0372		0004				3				0031			
4	03	0004	0003	0003				4		2455			2500		
5	04	0005	0003	0004				5		3716			2460		
6	01	0003	0004	0003				6		3746			2461		
7	04	0005	0003	0005				7		0001	0001				
0650	65	0003	0076	0003				0750						0372	
1	01	0003	0005	0005				1	03	2500	1766		2456		
2	17	0001	0002	0003				2	63	1506	3342		3530		
3	26	0001	0101	0002				3		2456			2511		
4	16	0003	0002	0002				4	03	2510	2510		2442		
5	05	0005	0002	0002				5	03	1311			2442		
6	36	0346	0001	0371				6			1766				
7	33		0010					7		2456			2460		
0660	37							0760	01	3000					
1		1473	2600					1	03	2457			2442		
2	01	1000						2		1765					
3		1410						3		1311					
4		1111	1040					4	03		2442		2442		
5	77	0034	1400					5	03	2510		2511	2442		
6	01	1777	3777	3777				6		0002					
7		0001		0001				7	04	0373	0002				
0670				0030				0770	03		2511				
1								1					3777		
2		0001						2		1766			3665		
3		3715		2453				3	30	0410			0372		
4		3745		2454				4	02	0051	0043		0052		
5		0031						5	04	0052	0051		0050		
6	40							6					2445		
7	03	1444	0773	1242				7	04	0414	0413		0414		
0700	72	1217	1341	1075				1000	22	0510	0400		0510		
1	54	1524	2377	1607				1	22	2445	0401		2445		
2	06	1624	2734	0260				2	75	2445	0402		0510		
3	07	1320						3					2445		
4	11	1034						4		0414			0001		
5	00							5	77				0060		
6	10	1320						6	03	0003	0050		2452		
7	10	1604						7	03	0002	0052		2453		
0710	06	1320						1010		2452			3716		
1	11	1166						1		2453			3746		
2	10	1034						2					2446		
3	11	1224						3					0001		
4	10	1130						4	77				0165		
5	07	1700						5	32	0002					
6	11	1130						6	22	0524	0403		0524		
7	06	1700						7	22	2446	0401		2446		

/138

Address		Command				Address		Command			
1020		75	2446	0444	0524	1120			0467		2451
1		72				1			0470		0646
2		22	0515	0403	0515	2			0403		2452
3		22	0521	0401	0521	3			0447		2453
4		22	0522	0401	0522	4	22	2450	2447		2454
5			0445		0524	5					2455
6		22	2445	0401	2445	6	15				2456
7		75	2445	0402	0515	7	16	2451	2454		0641
1030			0403		2445	1130	00				
1			0403		2446	1	01	2456	2442		2456
2			0403		2447	2	22	2455	0403		2455
3		22	2445	0403	2445	3	75	2455	2446		0654
4		77			0215	4	03	2456	0471		2456
5			0407	0410	0411	5	00				
6			0412	3777		6	22	0645	0401		0646
7		34		0611	0562	7	22	2446	0403		2446
1040			3777		1312	1140	22	2451	0447		2451
1		22	0551	0401	0551	6	75	2446	2443		0635
2		22	2447	0403	2447	2	34				0657
3		75	2447	2445	0545	3	22	2454	2452		2454
4		22	0563	0403	0563	4	22	2451	2453		2451
5		22	0564	0403	0564	5	34				0640
6		22	2446	0403	2446	6		0403			2446
7		75	2446	0406	0543	7		0373			2510
1050		34			0612	1150					2450
1					0600	1	22	2450	2446		2450
2			3716		2453	2	35	2446	2444		0666
3			3746		2454	3	22	2446	0403		2446
4		77			0060	4	34				0662
5		02	0003	2453	2455	5	22	0472	2444		2446
6		03	2455	2455	2455	6	22	2450	0473		2451
7		03	0002	0051	2457	7	26	2451	0113		2451
1060		02	2457	2454	2457	1160		0403			2452
1		03	2457	2457	2457	1		2451			2453
2		01	2455	2457	2455	2		2446			2454
3			0001		2452	3	26	2445	0113		2455
4			2455		0001	4					2457
5		77			0131	5	15				2456
6		25	0002	0077	2455	6	16	2453	2454		0700
7		00				7	00				
1070			2455		2456	1170	01	2456	2442		2456
1		22	0404	2447	0603	1	22	2457	0403		2457
2		00				2	75	2457	2452		0712
3		22	0405	2447	0605	3	00				
4		00				4	22	0704	0401		0704
5			2452		0001	5	22	2452	0403		2452
6		34		0600	0565	6	62	2451	0447		2451
7		03	2455	2456	0001	7	75	2452	2443		0672
1100		00				1200	34				0716
1			1312		0001	1	62	2454	0403		2454
2		77			0343	2	62	2453	2455		2453
3		04	0373	0002	1766	3	62	2455	0447		2455
4		22	0403	0403	2443	4	34				0677
5			0464		0704	5	22	2450	2443		2450
6			0465		0726	6	22	2450	0474		2446
7		62	2443	0403	2444	7		2444			2451
1110		62	2444	0403	2445	1210		2443			2452
1			0403		2446	1		0475			2453
2			0403		2447	2					2454
3					2450	3	15				2455
4		22	2450	0403	2450	4	16	2446	2453		0727
5		22	2447	2450	2447	5	00				
6		75	2450	2444	0625	6	00				
7			0466		2450	7	01	2455	2442		2455

/139

Address					Command					Address					Command				
1220		22	2454	0403	2454	1320				0454					2465				
1		35	2454	2443	0737	1				0450					2464				
2		22	0726	0460	0726	2									2446				
3		62	2446	2452	2446	3				0407					2454				
4		62	2452	0403	2452	4				2464					1057				
5		34			0725	5				0456					1046				
6		22	2447	2445	2447	6				0457					1047				
7		22	2447	0474	2446	7									2445				
1230			0476		2454	1330				0407					2455				
1	15				2452	1				2454					0001				
2					2456	2			77						0060				
3			2454		0746	3				0002					2456				
4	16		2446	2453	0750	4				0003					2457				
5	00					5				3716					2460				
6	25	2442	0001	2442		6				3746					2461				
7	00					7			02	2457	2460				2460				
1240	01	2452	2442	2452	1340	03			03	2460	2460				2460				
1	22	2456	0403	2456	1	03			03	0051	2456				2462				
2	35	2456	2451	0757	2	02			02	2462	2461				2461				
3	22	0746	0447	0746	3	03			03	2461	2461				2461				
4	62	2446	0403	2446	4	01			01	2460	2461				0001				
5	34				0745	5			77						0131				
6	22	2454	0460	2454	6	03			03	0002	3212				2462				
7	62	2446	0477	2446	7	01			01	2455	2462				2455				
1250	62	2451	0403	2451	1350	22			22	1046	0403				1046				
1	75	2451		0743	1	22			22	1047	0403				1047				
2	22	2450	0473	2450	2	22			22	1057	0447				1057				
3	26	2450	0126	2453	3	22			22	2445	0401				2445				
4		0500		2451	4	75			75	2445	2463				1046				
5	16	2453	2451	0771	5	25			25	2455	0077				2455				
6	01	2455	2452	0001	6					2455					2500				
7	77			0343	7	22			22	1067	0401				1067				
1260	00				1360	01			01	2454	0451				2454				
1		0403		2446	1	36			36	2454	0452				1035				
2		0501		2452	2	30			30	1403					0371				
3	22	0473	2447	2451	3	31			31		2500				2471				
4	26	2451	0126	2451	4	20			20	2471	0446				3072				
5	16	2452	2451	1000	5	30			30	1410					0372				
6	16	1000	2450	1000	6	31			31		2500				2470				
7	00				7	30			30	0410					0372				
1270	22	2446	0403	2446	1370	71			71						2470				
1	35	2446	2443	1005	1	35			35	2470	0446				1105				
2	22	1000	0502	1000	2	33			33	0011									
3	34			1000	3	34			34						1073				
4	22	2443	0403	2443	4	22			22	1076	0453				1076				
5	75	2443	0406	0616	5	22			22	1100	0453				1100				
6				2443	6	62			62	2463	0401				2463				
7		0444		2444	7	62			62	2465	0447				2465				
1300		0403		2445	1400	22			22	2464	2465				2464				
1		0503		1014	1					0455					1067				
2				2446	2	22			22	2446	0401				2446				
3	00				3	75			75	2446	0402				1034				
4	22	2446	0403	2446	4					0504					2453				
5	75	2446	2445	1026	5					0402					2455				
6	22	1014	0403	1014	6										2445				
7	62	1014	2444	1014	7					2453					2457				
1310	22	2445	0403	2445	1410										2446				
1	22	2444	0444	2444	1	30			30	0410					0372				
2	22	2443	0401	2443	2	31			31		2500				2470				
3	75	2443	0402	1013	3	35			35	2470	0446				1127				
4	34			1030	4	33			33	0012									
5	22	1014	0400	1014	5	34			34						1122				
6	34			1014	6					2457					1130				
7		0402		2463	7	00			00										

/140

Address		Command				Address		Command			
1420		31		1766	2470	1520			0002		
1		35	2470	0446	1135	1			0001		0001
2		33	0013			2		72	0054		0257
3		34			1130	3		00			
4					2454	4		31		0626	3777
5					2452	5		35	0046	3777	0264
6		03	2500	1766	2456	6		33	0016		
7		01	2456	2454	2454	7		34			0257
1430		22	1137	0460	1137	1530		15			0504
1		22	2452	0401	2452	1			0504		0001
2		75	2452	0461	1137	2		77			0060
3		03	2454	0451	2454	3			0003		0020
4			0462		1137	4		03	0002	0051	0021
5		22	1130	0453	1130	5		34			0626
6		22	2446	0401	2446	6			0001		1767
7		35	2446	0401	1157	7		01	0504	0220	0504
1440		76	2454	0463	1166	1540		22	0272	0222	0272
1		72	2454		0001	1		36	0504	0221	0265
2		77			0165	2		30	0410		0372
3		32	0002			3		31		2500	3777
4		33	0014			4		35	3777	0223	0303
5		34			1166	5		33	0017		
6		02	0127	2454	2454	6		34			0276
7		76	2454	0463	1166	7					0505
1450		72	2454		0001	1550		03	1767	2500	0504
1		77			0165	1		01	0504	0505	0505
2		32	0002			2		22	0304	0224	0304
3		32	2446			3		75	0304	0225	0304
4		33	0015			4		03	0220	0505	0440
5		75	2446	2455	1130	5		22	0276	0236	0276
6		22	1122	0453	1122	6			0226		0304
7		22	2445	0401	2445	7		22	0310	0222	0310
1460		62	2455	0401	2455	1560		75	0310	0227	0276
1		22	2453	0453	2453	1					0506
2		75	2445	0402	1120	2					0504
3		34			0034	3					0507
4		73	1467	2145	2625	4					0510
5		03	1441	0131	0455	5		03	3212	0440	0511
6					0001	6		01	0511	0510	0510
7		77	3777	3777	3777	7		22	0321	0231	0321
1470			0001	0001		1570		62	0231	0232	0231
1		03	2361	3072	0504	1		22	0507	0222	0507
2		03	1767	2500	0504	2		75	0507	0234	0321
3		03	0220	0505	0470	3		72	0510		3666
4			0030	0001		4			0510		0001
5			0030	0001		5		77			0165
6			0001			6		32	0002		
7					0030	7			0235		0321
1500					0030	1600		22	0504	0232	0504
1		03	3212	0440	0511	1		22	0321	0504	0321
2			0373			2			0230		0231
3		30	0400	2462		3		62	0234	0222	0234
4					0002	4		22	0327	0222	0327
5		34			0433	5		22	0506	0222	0506
6		34			0365	6		75	0506	0233	0317
7			0500		0001	7		35		0045	0356
1510			0626		0500	1610		22	0237	0055	0346
1			0627		0501	1					3777
2			0001			2		00			
3				0001		3		31		0626	3776
4					0003	4		35	3776	0223	0353
5		02	3716	0500	0505	5		33	0020		
6		02	3746	0501	0506	6		34			0346
7		03	3666	0506	0506	7		22	0346	0057	0346

/141

Address		Command				Address		Command			
1620		22	3777	0222	3777	1650		22	0374	0232	0374
1		34			0365	1		22	0401	0232	0401
2					0363	2		22	0504	0222	0504
3		35	0053		0433	3		75	0504	0233	0372
4			0053		0055	4					0507
5			0242		0353	5			0500		0001
6			0241		0427	6		77			0165
7		17	0346	0434	0237	7		32	0002		
1630		34			0344	1660		22	0411	0232	0411
1					0503	1		22	0507	0222	0507
2			0626		0500	2		75	0507	0250	0411
3			0627		0501	3		22	0503	0240	0503
4					0504	4		22	0366	0254	0366
5					0502	5		22	0367	0254	0367
6		02	3716	0500	0505	6		72	0251		0372
7		03	0505	0505	0505	7			0252		0374
1640		02	3746	0501	0506	1670			0253		0401
1		03	0506	0506	0506	1			0243		0411
2		01	0506	0505	0001	2		75	0503	0055	0366
3		77			0131	3			0244		0366
4		03	0002	0161	0506	4			0245		0367
5		03	3666	0506	0506	5		75	3777	0045	0346
6		01	0506	0502	0502	6		34			0357
7		22	0372	0232	0372	7		33	0021		
						1700		77	3777	3777	

/142

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